Deformations of the identity by unital, symmetric completely positive maps appear in important invariants of finite von Neumann algebras such as property (T) and the Haagerup property. Many examples of such deformations are also strongly continuous semigroup deformations, and such a deformation has a generator given by the ‘derivative at zero’, which is a densely-defined and closed nonnegative self-adjoint operator. Such a semigroup of completely positive maps on a finite von Neumann algebra is called a symmetric completely Markovian semigroup. The generator of a semigroup of positive maps on a finite von Neumann algebra can be used naturally to construct a quadratic form on the GNS Hilbert space $L^2$ obtained from the trace. The properties of the semigroup being symmetric and completely Markov are reflected in the resulting quadratic form as a complete Lipschitz property, which is the "amplified" version of the following Lipschitz property: The form is real, and on the real part of the domain of the form, the value of the form decreases when applying any Lipschitz function that vanishes at zero. Completely Lipschitz forms are also completely Dirichlet, in that they satisfy the amplified version of the following Dirichlet property: Viewing vectors in $L^2$ as affiliated operators, the domain of the form is stable under taking absolute value and the form sends the absolute value of any element in the domain to no greater than twice the form evaluated at the element. The Dirichlet property is important because if a form is Dirichlet, then the intersection of its domain with the von Neumann algebra is itself a dense $*$-subalgebra. Such subalgebras are called Dirichlet algebras. The divergence of a closable, real bimodule-valued derivation whose domain contains a Dirichlet algebra is the generator of a Dirichlet form. Conversely, given a Dirichlet form one can construct a ‘carre du champ’ that may be used to construct a right module such that there is a unique left module structure that makes the map $a \rightarrow a \otimes 1$ a real derivation defined on the domain of the (square root of the?) Dirichlet form. By Corollary 3.5 in Sauvageot 1, it’s evident that if a Dirichlet form is generated by the divergence of a real closable derivation then the semigroup generated by this divergence is in fact comprised of weakly continuous completely positive normal contractions (are they symmetric due to the reality of the derivation...and Markovian?) It should follow that the form was, in fact, completely Dirichlet. If this is so, and the processes of
going between the form and the derivation are inverses, then every Dirichlet form is completely Dirichlet, and that can’t be right...

2. Introduction

This paper aims to collect in one place the known interrelations between strongly continuous semigroups of completely positive maps (on a finite von Neumann algebra $N$), quadratic forms on Hilbert space (which one?), and $N$-bimodule-valued derivations found in $[]$, $[]$, $[]$. As an application of these ideas, we include at the end a proof of a von Neumann algebra analogue of a result of Bekka and Valette $[]$. The proof of this new result uses the interrelations presented in an essential way, and is a completely different approach from that of the original proof of Bekka and Valette’s result.

2.1. Basic Concepts and Notation. We attempt to adhere to standard notation as much as possible in what follows. In this paper, $N$ will denote a finite von Neumann algebra with normal tracial state $\tau$. We denote by $L^2(N) = L^2(N, \tau)$ the GNS Hilbert space obtained using $\tau$, and denote by $\Omega$ the unit cyclic vector in $L^2(N)$ that is the image of the multiplicative identity element $1 \in N$.

3. Completely Positive Maps and Deformations

For more details on completely positive maps in general, the reader is referred to [Paulsen], whom we follow very closely in this section.

3.1. Completely Positive Maps on Finite von Neumann Algebras. A map $\varphi : N \to N$ is positive if it preserves the positive cone of $N$, meaning that $\varphi(x) \geq 0$ for all $x \geq 0$ in $N$. Let $M_k(N)$ denote the von Neumann algebra of $k \times k$ matrices $(x_{ij})$ with entries $x_{ij} \in N$ acting in the obvious way on column vectors in $L^2(N)^{\otimes k}$. The map $\varphi$ naturally induces an inflation map $\varphi^{(k)} : M_k(N) \to M_k(N)$ that applies $\varphi$ entrywise:

$$\varphi^{(k)}((x_{ij})) = (\varphi(x_{ij})).$$

The map $\varphi$ is completely positive (c.p.) if all of the maps $\varphi^{(k)}$ are positive.

A completely positive map from $N$ into itself is unital if it fixes 1, and normal if it is a continuous self-map on $N$ equipped with the weak operator topology. In these notes, we’ll most often consider unital normal completely positive maps. Unless otherwise specified, in what follows ‘completely positive map on $N$’ will mean ‘unital normal completely positive map from $N$ into itself’.

A completely positive map $\varphi$ on $N$ is symmetric with respect to $\tau$ if for all $x, y \in N$

$$\tau(\varphi(x)y) = \tau(x\varphi(y)).$$

and are $\tau$-preserving if $\tau(\varphi(x)) = \tau(x)$ for all $x \in N$. It is clear that, since they are assumed unital, symmetric completely positive maps are all $\tau$-preserving.

Before moving on, we note three basic examples of completely positive maps.
Example 3.1. Every unital, normal *-automorphism $\alpha$ of $N$ is a completely positive map on $N$. To see this, recall that $x \geq 0$ in $N$ if and only if there is a $y \in N$ such that $y^* y = x$. Complete positivity of $\alpha$ follows, since $\alpha((y_{ij})^*)(y_{ij})) = (\alpha(y_{ij}))^*(\alpha(y_{ij}))$.

Example 3.2. (See section 3.6 of [SiSm]) When $1 \in B \subset N$ is an inclusion of finite von Neumann algebras, the orthogonal projection of $L^2(N) = \overline{\mathcal{N}}$ onto $\overline{\mathcal{B}}$ restricts uniquely to a map from $N \Omega$ onto $\overline{B \Omega}$ that may be regarded as a normal linear map from $N$ onto $B$, called the normal conditional expectation $E_B$. This map is completely positive, which is straightforward to verify using the above description of $E_B$ as the ‘restriction’ of a self-adjoint projection to show that for every $(x_{ij}) \in M_n(N)_+$ it is true that $\langle E_B^{(n)}((x_{ij})), \xi, \xi \rangle = \langle (x_{ij}), \xi, \xi \rangle \geq 0$ for all $\xi \in B \Omega^{\overline{n}}$. Positivity of $E_B^{(n)}$ follows since $B \Omega^{\overline{n}}$ is dense in the Hilbert space $\overline{B \Omega^{\overline{n}}}$. The above description also reveals that $E_B$ is always a symmetric completely positive map.

Example 3.3. (See Haagerup Lemma 1.1 of Non-Nuc C* alg w/ approx prop) We now consider the important fact that any positive definite function $\phi$ on a countable discrete group $\Gamma$ gives rise to a completely positive map on the group von Neumann algebra $L\Gamma$, and vice-versa. Recall that a function $\phi : \Gamma \to \mathbb{C}$ on a countable discrete group $\Gamma$ is positive definite if for any finite set $\{g_1, ..., g_n\} \subseteq \Gamma$ the matrix $(\phi(g_j^{-1} g_i)) \in M_n(\mathbb{C})$ is positive semidefinite Hermitian, i.e. $\phi(g) = \overline{\phi(g^{-1})}$ for all $g \in \Gamma$ and for all finite subsets $\{c_1, ..., c_n\} \subseteq \mathbb{C}$

$$\sum_{i,j} c_i \overline{c_j} \phi(g_j^{-1} g_i) \geq 0.$$  

In order to have a fundamental example of a positive definite function to keep in mind, we note that for every $t > 0$ the map $\phi_t$ on the free group $\mathbb{F}_n$ is completely positive, where $\phi_t(g) = e^{-|t|g}$, where $|g|$ is the word-length with respect to the standard generators of $\mathbb{F}_n$ (see [Haagerup]).

It is well-known that a positive definite function $\phi$ on $\Gamma$ can be used to construct a cyclic unitary representation $(\pi_\phi, \mathcal{H}_\phi, \Omega_\phi)$ of $\Gamma$ via a GNS construction, and that from the representation $(\pi_\phi, \mathcal{H}_\phi, \Omega_\phi)$ we recover $\phi$ as $(\pi_\phi(\cdot) \Omega_\phi, \Omega_\phi)$. If $\Gamma$ is a countable discrete group and $\phi$ is a positive definite function on $\Gamma$, then the representation is a completely positive map $\Phi$ on the group von Neumann algebra $L\Gamma$ such that $\Phi(\sum \lambda g L_g) = \sum \lambda \phi(g) L_g$ on finite sums. To see this, let $(\pi_\phi, \mathcal{H}_\phi, \Omega_\phi)$ be the cyclic representation of $\Gamma$ obtained from $\phi$. Let $(e_i)$ be an orthonormal basis for $\mathcal{H}_\phi$ and define

$$a_i(g) = \langle \pi_\phi(g) e_i, \Omega_\phi \rangle$$

for all $g \in \Gamma$. Since each $\pi_\phi(g)$ is a bounded operator, every $a_i \in L^\infty \Gamma$. For each of the $a_i$, associate the multiplication operator $M_{a_i}$ on $l^2 \Gamma$, i.e. the operator $\xi \mapsto M_{a_i} \xi$ such that $(M_{a_i} \xi)(g) = a_i(g) \xi(g)$ for all $g \in \Gamma$. Trivial scratchwork shows that $M_{a_i}^* = M_{\overline{a_i}}$. Now, to each $x \in L\Gamma$ associate the bounded operator $P x = \sum_i M_{a_i} x M_{a_i}^*$ on $l^2 \Gamma$. The map $x \mapsto P x$ is easily seen to be weak-operator continuous, and is completely positive since it is a finite sum of maps of the form $x \mapsto M_{a_i} x M_{a_i}^*$, the inflations of which are conjugations by $\text{diag}(M_{a_i})$, hence are all positive. One also
readily checks that \((\langle PL_0|\xi\rangle(h) = (\sum_i a_i(h)a_i(\overline{g^{-1}h}))\overline{\xi(h^{-1}g)}\) for every \(h \in \Gamma\). Finally, 
\[
\sum_i a_i(h)a_i(\overline{g^{-1}h})\overline{\xi(h^{-1}g)} = \phi(g)(L_g|\xi\rangle(h) \text{ as verified by the computation } \\
\sum_i a_i(h)a_i(\overline{g^{-1}h}) = \sum_i \langle \pi_\phi(h)e_i, \Omega_\phi \rangle \langle \Omega_\phi, \pi_\phi(g^{-1}h)e_i \rangle \\
= \sum_i \langle e_i, \pi_\phi(h^{-1})\Omega_\phi \rangle \langle \pi_\phi(\overline{g^{-1}h}), \pi_\phi(h^{-1})\Omega_\phi, e_i \rangle \\
= \langle \pi_\phi(\overline{g^{-1}h})\Omega_\phi, \pi_\phi(h^{-1})\Omega_\phi \rangle \\
= \langle \pi_\phi(h^{-1})\pi_\phi(g)|\Omega_\phi, \pi_\phi(h^{-1})\Omega_\phi \rangle \\
= \langle \pi_\phi(g)|\Omega_\phi, \Omega_\phi \rangle = \phi(g).
\]

In the other direction, suppose that \(\Phi : L\Gamma \to L\Gamma\) is a completely positive map, then if \(u_g\) is the unitary operator in \(L\Gamma\) corresponding to the group element \(g\), then the map \(g \mapsto \tau(u_g^*\Phi(u_g))\) is a positive definite function on \(\Gamma\). Positive definiteness of this map is a direct consequence of the complete positivity of \(\Phi\). The condition \(\tau(u_g^*\Phi(u_g)) = \tau(u_{g^{-1}}^*\Phi(u_g^{-1}))\) for all \(g \in \Gamma\) follows from (1) \(u_{g^{-1}} = u_g^*\), (2) \(\Phi(u_g^*) = \Phi(u_g)^\ast\), which follows immediately from the decomposition \(u_g = (h_+ - h_-) + i(k_+ - k_-)\) with \(h_\pm, k_\pm \in L\Gamma_\pm\), and (3) \(\tau\) is a Hermitian linear functional: \(\tau(x^*) = \tau(x)\) for all \(x \in L\Gamma\), and (4) \(\tau\) is tracial.

The standard example of a positive map that is not completely positive is the transpose map on \(M_2(\mathbb{C})\), which is a fact easy to discover by considering that \((a_{ij}) \geq 0\) in \(M_k(\mathbb{C})\) if and only if \(\langle (a_{ij})\xi, \xi \rangle \geq 0\) for all \(\xi \in \mathbb{C}^k\).

The following theorem of Stinespring characterizes completely positive maps on \(N\) as ‘compressions’ of unital, normal \(*\)-homomorphisms into a larger \(B(\mathcal{H})\).

**Theorem 3.4.** (Stinespring dilation theorem) Suppose \(\varphi\) is a completely positive map on \(N\). Then there exists a Hilbert space \(\mathcal{H}\), a normal unital \(*\)-representation \(\pi : N \to B(\mathcal{H})\), and an isometry \(V : L^2(N) \to \mathcal{H}\) such that for all \(x \in N\)
\[
\varphi(x) = V^*\pi(x)V.
\]

**Proof.** Consider the sesquilinear form on the vector space \(\mathcal{H}_0 = N \otimes_{\text{alg}} L^2(N)\) defined by linearly extending
\[
\langle x \otimes \xi, y \otimes \eta \rangle_{\mathcal{H}_0} = \langle \varphi(y^*x), \xi, \eta \rangle_{L^2(N)}.
\]

Positive semidefiniteness of this form follows immediately from complete positivity of \(\varphi\) since for every \(n\)
\[
\langle \sum_{j=1}^n x_j \otimes \xi_j, \sum_{i=1}^n x_i \otimes \xi_i \rangle_{\mathcal{H}_0} = \langle \varphi^{(n)}((x_i^*x_j)), \xi_1, \ldots, \xi_n \rangle_{L^2(N)^{\otimes n}} \geq 0.
\]

By the Cauchy-Schwartz inequality the set \(\mathcal{K} = \{ \eta \in \mathcal{H}_0 : \langle \eta, \eta \rangle_{\mathcal{H}_0} = 0 \}\) is a subspace of \(\mathcal{H}_0\) and we may form the quotient inner product space \(\mathcal{H}_0/\mathcal{K}\) and take its
completeness to obtain a Hilbert space $\mathcal{H}$. By a matrix factorization, the inequality of matrices
\[(x_i y^* y x_j) \leq \|y^* y\| (x_i^* x_j)\]
holds in the positive cone $M_k(N)^+ \subset M_k(N)$. Using this inequality one readily verifies that for every $y \in N$ the map $\pi(y) : \mathcal{H}_0 \to \mathcal{H}_0$ defined by
\[\pi(y)(\sum_{j=1}^n x_j \otimes \xi_j) := \sum_{j=1}^n y x_j \otimes \xi_j,\]
extends to a bounded operator on $\mathcal{H}$, which we still call $\pi(y)$, and furthermore $\pi : N \to B(\mathcal{H})$ is a normal unital $*$-homomorphism. The map $V : L^2(N) \to \mathcal{H}$ defined by $V(\xi) = 1 \otimes \xi + \mathcal{K}$ is an isometry since $\varphi$ is unital. Given $x \in N$, for every $\xi, \eta \in L^2(N)$ one checks the equality of matrix coefficients
\[\langle V^* \pi(x)V \xi, \eta \rangle_{L^2(N)} = \langle x \otimes \xi, 1 \otimes \eta \rangle_{\mathcal{H}} = \langle \varphi(x) \xi, \eta \rangle_{L^2(N)}\]
so $V^* \pi(x)V = \varphi(x)$. \qed

It is an easy exercise to show that all maps of the form $V^* \pi(x)V$ with $\pi$ and $V$ taken as in the statement of Stinespring’s theorem above are completely positive maps on $N$, so Stinespring’s theorem characterizes all completely positive maps on $N$. Note that if we identify $L^2(N)$ with its image $VL^2(N)$ in $\mathcal{H}$ then $V^*$ becomes the projection $P$ of $\mathcal{H}$ onto $L^2(N)$ so this identification provides the intuition needed to think of $\varphi(x)$ as $P \pi(x)|_{\mathcal{H}}$, for every $x \in N$.

It will be helpful in what follows to have the property considered in the following

\textbf{Proposition 3.5.} A $\tau$-preserving completely positive map $\varphi$ induces a contraction on $L^2(N)$.

\textit{Proof.} Suppose $\varphi$ is completely positive, and consider the Stinespring dilation $\varphi = Ad(V) \circ \pi$, then
\[\varphi(y)^* \varphi(y) = (V^* \pi(y)^*)VV^* \pi(y)V \leq (V^* \pi(y)^*)1 \pi(y)V = V^* \pi(y^* y)V = \varphi(y^* y).\]

Appealing to the linearity and positivity of the trace, the above inequality is easily used to deduce the result. \qed

3.2. von Neumann Algebra Properties Expressable in terms of deformations by CP Maps.

3.3. Examples.

4. Correspondences

4.1. Examples (Particularly from CP maps).

4.2. Properties Expressed in terms of Correspondences.
5. Semigroup Deformations

Since the deformations we consider are by unital, tracial completely positive maps, the maps \( \varphi_t \) induce bounded operators \( \Phi_t \) on \( B(L^2(N)) \). A deformation \((\varphi_t)_{t \geq 0}\) is a semigroup deformation if the net \((\Phi_t)_{t \geq 0}\) of induced operators is a one-parameter semigroup of operators on \( L^2(N) \) as discussed below. We follow Chapter 13 of [Rudin’s Functional Analysis] closely here.

In this section, let \( \mathcal{H} \) denote a Hilbert space, and for every \( t \geq 0 \) suppose there is associated an operator \( Q_t \in B(\mathcal{H}) \) such that

1. \( Q_0 = 1 \),
2. \( Q_{s+t} = Q_s Q_t \) for all \( s \geq 0 \) and \( t \geq 0 \),
3. \( \lim_{t \to 0} ||Q_t \xi - \xi|| = 0 \) for every \( \xi \in \mathcal{H} \). (strong continuity)

Then \((Q_t)_{t \geq 0}\) is called a strongly continuous one-parameter semigroup of operators, or simply a semigroup.

Motivated by the fact that every continuous complex function \( f : [0, \infty) \to \mathbb{C} \) satisfying \( f(s + t) = f(s) + f(t) \) for all \( s, t \geq 0 \) has the form \( f(t) = \exp(\alpha t) \) and is determined by the number \( f'(0) \), we associate with \((Q_t)_{t \geq 0}\) the operators \( A_\varepsilon \), where

\[
A_\varepsilon \xi = \frac{1}{\varepsilon} [Q_\varepsilon \xi - \xi]
\]

for all \( \xi \in \mathcal{H} \) and \( \varepsilon > 0 \). We then define the operator \( A \) on \( \mathcal{H} \)

\[
A \xi = \lim_{\varepsilon \to 0} A_\varepsilon \xi
\]

for all \( \xi \). The right hand limit is of course a limit of vectors in \( \mathcal{H} \) and the domain \( D(A) \) of \( A \) is the set of those \( \xi \in \mathcal{H} \) for which the limit exists. It is clear that \( D(A) \) is a subspace of \( \mathcal{H} \), and that \( A \) is a linear operator on \( \mathcal{H} \). The operator \( A \) is analogous the the derivative at zero and is called the infinitesimal generator of the one-parameter semigroup \((Q_t)_{t \geq 0}\). In what follows, we collect the main results about semigroups from Chapter 13 of [Ru]. We include more than we need in what follows, for future reference.

**Theorem 5.1.** If the semigroup \((Q_t)_{t \geq 0}\) satisfies the preceding hypotheses, then

\( a \): \( t \mapsto Q_t \xi \) is a continuous map of \([0, \infty)\) into \( \mathcal{H} \), for every \( \xi \in \mathcal{H} \),

\( b \): \( A \) is a closed densely defined operator on \( \mathcal{H} \),

\( c \): for every \( \xi \in D(A) \), \( Q_t \xi \) satisfies the differential equation

\[
\frac{d}{dt} Q_t \xi = AQ_t \xi = Q_t A \xi,
\]

\( d \): for every \( \xi \in \mathcal{H} \),

\[
Q_t \xi = \lim_{\varepsilon \to 0} [\exp(tA_\varepsilon)][\xi],
\]

the convergence being uniform on all compact subsets of \([0, \infty)\).

**Proof.** (a) If there were a sequence \( t_n \to 0 \) such that \( \|Q_{t_n}\| \to \infty \) then there must exist \( \xi \in \mathcal{H} \) such that \( \|Q_{t_n} \xi\| \to \infty \) by the Banach-Steinhaus Theorem. This contradicts
the strong continuity assumption. It follows that there is an $M > 0$ such that $||Q_t|| \leq M$ for all $t \geq 0$. Given $\xi \in \mathcal{H}$ and a net $t_\lambda \to t$ we have

$$
||Q_t - Q_t(x)|| = ||Q_{t_\lambda} - Q_t(x)||
= ||Q_t(Q_{t_\lambda} - 1)||
\leq ||Q_t|| ||Q_{t_\lambda} - 1||
\leq M ||Q_{t_\lambda} - 1|| \to 0
$$

by strong continuity.

(b) To show that the operator $A$ is densely defined, we prove that for any element $\xi \in D(A)$ there is a net of elements in $D(A)$ converging to $\xi$. In order to do this, we note that the continuity in (a) allows us to define operators $M_t (t \geq 0)$ such that $M_t \xi_0 \to \xi_0$ for all $\xi_0 \in \mathcal{H}$ and $A_t M_\varepsilon = A_\varepsilon M_t$ for every $\varepsilon > 0$ and $t > 0$. From this relation it will follow that

$$
\lim_{\varepsilon \to 0} A_\varepsilon M_t \xi = \lim_{\varepsilon \to 0} A_t M_\varepsilon \xi = \lim_{\varepsilon \to 0} M_\varepsilon A_t \xi = A_t \xi
$$

establishing that $M_t \xi \in D(A)$ for every $t > 0$. The right choice of $M_t$ is

$$
M_t \xi = \frac{1}{t} \int_0^t Q_s \xi ds,
$$

where the integral exists (as a norm limit of Riemann sums) by the continuity established in (a). The identity $A_t M_\varepsilon = A_\varepsilon M_t$ for every $\varepsilon > 0$ and $t > 0$ is easily verified by a direct computation appealing to the semigroup law and integration by (easy) substitution. In the end, this amounts to verifying that the ‘not-quite-symmetrical’ integrals

$$
A_t M_\varepsilon \xi = \frac{1}{t \varepsilon} \int_0^t [Q_{t+s} - Q_t] \xi ds
$$

and

$$
A_\varepsilon M_t \xi = \frac{1}{t \varepsilon} \int_0^t [Q_{\varepsilon+s} - Q_s] \xi ds
$$

are equal. In fact, $A_\varepsilon M_t = M_t A_\varepsilon = M_\varepsilon A_t = A_t M_\varepsilon$ is easy to verify for any $\varepsilon, t > 0$ since $Q_s Q_t = Q_t Q_s$ for any $t, s > 0$. The fact that $M_t \xi \to \xi$ as $t \to 0$ is almost a direct consequence of strong continuity, because we may use $Q_t \xi \approx Q_0 \xi = \xi$ for small $t$ to approximate Riemann sums with sums in which all terms but $t \xi$ telescope away.

To show that $A$ is a closed operator, we use the following useful property of the $M_t$, which essentially is that $M_t$ can filter $A_t$ out of $A$ for any $\xi \in D(A)$ and $t > 0$:

$$
M_t A \xi = M_t \lim_{\varepsilon \to 0} A_\varepsilon \xi
= \lim_{\varepsilon \to 0} M_t A_\varepsilon \xi
= \lim_{\varepsilon \to 0} M_\varepsilon A_t \xi
= A_t \xi.
$$
Then, if $\xi_\lambda \in D(A)$ and $\xi_\lambda \to \xi$, and $A\xi_\lambda \to \eta$, we have that $A_\lambda \xi_\lambda \to A_\lambda \xi$ and also $A_\lambda \xi_\lambda = M_\lambda A \xi_\lambda \to M_\lambda \eta$ so that $A_\lambda \xi = M_\lambda \eta$. Since $M_\lambda \eta \to \eta$ as $t \to 0$ we have that $A_\lambda \xi = \lim_{\lambda \to 0} A_\lambda \xi$ exists and equals $\eta$. It follows that $A$ is closed.

(c) Let $\xi \in D(A)$. It is clear that $A_\varepsilon Q_t = Q_t A_\varepsilon$ for all $\varepsilon > 0$ and $t \geq 0$. Therefore

$$\lim_{\varepsilon \to 0} A_\varepsilon Q_t \xi = \lim_{\varepsilon \to 0} Q_t A_\varepsilon \xi = Q_t A_\lambda \xi,$$

and $Q_t \xi \in D(A)$ and $AQ_t \xi = Q_t A_\lambda \xi$. Now

$$\frac{d}{dt} Q_t \xi = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [Q_{t+\varepsilon} - Q_t] \xi = Q_t [\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} [Q_\varepsilon - 1] \xi] = Q_t A_\lambda \xi.$$

(d) This requires an estimate...see Rudin.

We also note the following result that characterizes those semigroups arising from exponentiating with a bounded generator.

**Theorem 5.2.** Consider $(Q_t)$ as above. Any of the following three conditions implies the other two:

- **a’:** $D(A) = H$,
- **b’:** $\lim_{\varepsilon \to 0} \|Q_\varepsilon - 1\| = 0$,
- **c’:** $A \in B(H)$ and $Q_t = e^{tA} (0 \leq t < \infty)$.

Our final result of this section is the only one that actually requires that $H$ be a Hilbert space. The two results above hold even if $H$ is replaced by a Banach space $X$.

**Theorem 5.3.** Assume that $(Q_t)_{t \geq 0}$ is a semigroup of normal operators $Q_t \in B(H)$, satisfying the continuity condition

$$\lim_{t \to 0} \|Q_t \xi - \xi\| = 0 \quad (\xi \in H).$$

The infinitesimal generator $A$ of $(Q_t)$ is then a normal operator in $H$, there is $\gamma < \infty$ such that $\Re \lambda \leq \gamma$ for every $\lambda \in \sigma(A)$, and

$$Q_t = e^{tA} \quad (0 \leq t < \infty).$$

If each $Q_t$ is unitary, then there is a self-adjoint operator $S$ on $H$ such that

$$Q_t = e^{i t S} \quad (0 \leq t < \infty).$$

(This representation of unitary semigroups is a classical theorem of Marshall Stone.)

5.1. Example (Group-like deformation).

5.2. Another Example.

5.3. Symmetric Competely Positive Semigroups.
5.3.1. Examples.

5.4. Generators of Strongly Continuous Semigroups of Positive Maps.

5.4.1. Examples. Mention at the end of this section that if the semigroups have additional properties, these properties are reflected in the generator.

6. Quadratic Forms

Let $\mathcal{H}$ denote a complex Hilbert space. A nonnegative quadratic form on $\mathcal{H}$ is a nonnegative real-valued map $Q$ defined on a dense subspace $D$ of $\mathcal{H}$ such that there is a positive, densely defined self-adjoint operator $\Delta$ on $\mathcal{H}$ satisfying $Q(\xi) = ||\Delta^{1/2}(\xi)||^2$ for all $\xi \in D$. In this paper, all quadratic forms considered will be nonnegative, and hence ‘quadratic form’ shall mean ‘nonnegative quadratic form’ from here on.

Polarization obtains a sesquilinear form $q: \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ from a quadratic form $Q$, and any nonnegative sesquilinear form $q$ naturally gives a quadratic form by ‘restricting to the diagonal’.

6.1. Real Quadratic Forms. See P. 388 [DL]

We may consider quadratic forms on $L^2(N, \tau) = L^2_{s.a.}(N, \tau) \oplus iL^2_{s.a.}(N, \tau)$ that have an extra symmetry. A quadratic form $Q$ on $L^2(N, \tau)$ is real if $D = D_{s.a.} \oplus iD_{s.a.}$ and the associated sesquilinear form $q$ is real-valued on $D_{s.a.}$.

Example 6.1. Real quadratic form

Example 6.2. Quadratic form that is not real

6.2. Closed Quadratic Forms and Lower Semicontinuity. P. 387 DL

Let $Q$ be a nonnegative quadratic form. The quadratic form norm on the domain $D$ of $Q$ is given by

$$||x||^2_Q = ||x||^2 + Q(x).$$

The form $Q$ is closed if it satisfies the following condition: For all $(x_n)$ in $D$ with $x_n \to x \in \mathcal{H}$, $||x_n - x_m||_Q \to 0$ implies $x \in D$ and $||x_n - x||_Q \to 0$.

It is a fact that $Q$ is closed if and only if $Q: \mathcal{H} \to [0, \infty]$ is lower semicontinuous, i.e. if $\liminf_{\xi \to \xi_0} Q(\xi) = Q(\xi)$ for $\xi, \xi_0$ in $D$. (DELICATE POINT HERE...check)

6.3. Closable Quadratic Forms. Let $Q$ be a nonnegative quadratic form. The form $Q$ is said to be closable if for all $(x_n)$ in $D$ such that $x_n \to 0$ in $\mathcal{H}$ and $||x_n - x_m||_Q \to 0$ it follows that $||x_n||_Q \to 0$. (Fill in why this implies the existence of a closed extension in a way analogous to closable operators...)

6.4. Lipschitz Forms. See pp. 391 and 392 of DL.

Let $\text{Lip}_0^C = \{ f: \mathbb{R} \to \mathbb{C}: |f(s) - f(t)| \leq |s - t| \text{ for all } s, t \in \mathbb{R}, f(0) = 0 \}$, and let $\text{Lip}_0$ be the set of real-valued functions in $\text{Lip}_0^C$.

An $\mathbb{R}$-valued quadratic form $Q$ on $L^2(N)$ with domain $D$ is Lipschitz if $Q$ is real, $f(x) \in D$ and $Q(f(x)) \leq Q(x)$ for all $x \in D_{s.a.}$ and $f \in \text{Lip}_0$.

If $Q$ is a closable Lipschitz form on $L^2(N)$ with domain $D$, then its closure $\overline{Q}$ is also Lipschitz.
6.5. **Dirichlet Forms.** P. 397 of DL

A densely defined quadratic form $Q$ on $L^2(N)$ is *n-Lipschitz* if it's $n$-fold inflation $Q^{(n)}$ is Lipschitz. A **Dirichlet form** on $L^2(N)$ is a Lipschitz form with domain $D$ satisfying the further condition: $x \in D$ implies $|x| \in D$ and $Q(|x|) \leq 2Q(x)$. 2-Lipschitz implies Dirichlet (P.398 DL).

**Proposition 6.3.** Let $Q$ be a quadratic form on $L^2(N)$. If $Q$ is 2-Lipschitz, then $Q$ is Dirichlet.

6.6. **Completely Dirichlet Forms.** P.397 of DL

6.7. **Dirichlet Algebras and $C^1$-Functional Calculus.** Domain of Dirichlet form is an algebra: P.398 DL

Definition of Dirichlet Algebra p. 82 of Cipriani-Sauvageot

$C^1$-functional calculus: Perhaps the lemma from DL, or p. 340 of Sauvageot 1 or p. 103 of Cipriani-Sauvageot.

7. **Correspondence-Valued Derivations**

Let $\delta : D_\delta \rightarrow \mathcal{H}$ indicate a derivation, where $D_\delta$ is a $*$-subalgebra of $N$ that contains 1 and is dense when viewed as a subspace of $L^2(N, \tau)$, and $\mathcal{H}$ is a correspondence of $N$.

7.1. **Closable Derivations.** The derivation $\delta$ is **closable** if it is closable as an operator from $L^2(N, \tau)$ into $\mathcal{H}$.

7.2. **Real Derivations.** See page 4 of Peterson 1-cohomology. The derivation $\delta$ is **real** if $\langle x\delta(y), \delta(z) \rangle = \langle \delta(z^*), \delta(y^*)x^* \rangle$ for all $x, y, z \in D_\delta$. Given a closable derivation, we can cook up real derivations using the adjoint derivation (see Peterson).

Another way to treat all of this may be to use symmetric derivations: See page 3 of Peterson’s lecture 1 on derivations IHP.

7.3. **Examples.**

8. **Completely Conditionally Negative Definite (CCND) Maps (?)**

9. **Property (T) in terms of Derivations (?)**

10. **Interrelations**

10.1. **From Quadratic Forms to Semigroups.**

10.1.1. **Lipschitz Forms yield Symmetric Markovian Semigroups.** P. 395 of DL (This should perhaps be grouped with the converse below!)

10.1.2. **n-Lipschitz Forms yield Symmetric n-Markovian Semigroups.** P. 397 of DL Completely Lipschitz Forms yield Symmetric Completely Markovian Semigroups. Immediately from the above line.

Q: Does Completely Dirichlet Yield Symmetric Completely Markovian Semigroup?

10.2. **From Semigroups to Quadratic Forms.**
10.2.1. **Obtaining the Generator of Strongly Continuous Semigroup of Positive Maps.** See Rudin Functional Analysis p.355 (Perhaps this subsection can be omitted!)

10.2.2. **Symmetric Markovian Semigroups Yield Lipschitz Forms.** P.395 of DL

10.2.3. **Symmetric Completely Markovian Semigroups yield Completely Lipschitz (hence Completely Dirichlet) Forms.** Immediate from the line above.

10.3. **From Derivations to Semigroups (or Quadratic Forms).** Perhaps can begin with a derivation whose domain contains an (abstract) Dirichlet algebra and build a Dirichlet form from it.

If not, just look at P.340 of Sauvageot 1

10.4. **From Semigroups (or Quadratic Forms) to Derivations.** Cipriani-Sauvageot treats this, but also it can be found on p.324-325 of Sauvageot 2.

Are any/all of these processes inverses of one another???

11. **Analogue of a Theorem of Bekka and Valette**

12. **Appendix A: Functional Calculus for Unbounded Operators**

In these notes, we emphasize use of the spectral theorem, rather than belabor its proof. Therefore we state results here and resist the temptation to give proofs. The statements come from Ch. 13 of Rudin’s Functional Analysis.

**Definition 12.1.** Let $\mathcal{M}$ be a $\sigma$-algebra in a set $\Omega$, let $\mathcal{H}$ be a Hilbert space. A map $E : \mathcal{M} \to B(\mathcal{H})$ is called a **resolution of the identity** if it satisfies the following properties:

- **a:** $E(\emptyset) = 0$, $E(\Omega) = 1$.
- **b:** Each $E(\omega)$ is a self-adjoint projection.
- **c:** $E(\omega' \cap \omega'') = E(\omega') E(\omega'')$.
- **d:** If $\omega' \cap \omega'' = \emptyset$, then $E(\omega' \cup \omega'') = E(\omega') + E(\omega'')$.
- **e:** For every $\xi \in \mathcal{H}$ and $\eta \in \mathcal{H}$, the set function $E_{\xi,\eta}$ defined by

  $$E_{\xi,\eta}(\omega) = \langle E(\omega)\xi, \eta \rangle$$

  is a complex measure on $\mathcal{M}$.

Suppose that $E$ is a given resolution of the identity. For any complex $\mathcal{M}$-measurable function $f$ on $\Omega$, the essential range of $f$ is the smallest closed subset of $\mathbb{C}$ that contains $f(p)$ for almost all $p \in \Omega$ (i.e. the set contains all $f(p)$ but those in some set $\omega \in \mathcal{M}$ such that $E(\omega) = 0$). Let $L^\infty(E)$ denote the Banach algebra of all essentially bounded measurable functions (modulo the closed ideal of those functions $f$ with $||f||_\infty = 0$).

**Theorem 12.2.** If $E$ is a resolution of the identity as above, then the formula

$$\langle \Psi(f)\xi, \eta \rangle = \int_\Omega f dE_{\xi,\eta}, \quad (\xi \in \mathcal{H}, \eta \in \mathcal{H})$$
defines an isometric isomorphism $\Psi$ of the Banach algebra $L^\infty(E)$ onto a (not necessarily unital) abelian $C^*$-subalgebra $A$ of $B(H)$. This isomorphism is also a $*$-isomorphism: $\Psi(f) = \Psi(f)^*$ for all $f \in L^\infty(E)$. Furthermore,

$$||\Psi(f)\xi||^2 = \int \Omega |f|^2 dE_{\xi,\xi} \quad (\xi \in H, \ f \in L^\infty(E)).$$

Finally, an operator $Q \in B(H)$ commutes with every $E(\omega)$ if and only if $Q$ commutes with every $\Psi(f)$, i.e. $Q \in A'$.

The spectral theorem asserts that every bounded normal operator $T$ on a Hilbert space $H$ canonically induces a resolution of the identity $E$ on the Borel subsets of the spectrum $\sigma(T)$, and that $T$ can be reconstructed from $E$ by an integral of the above type. We exhibit the spectral theorem as a corollary of the following theorem.

**Theorem 12.3.** If $A$ is a unital $C^*$-subalgebra of $B(H)$ and $K$ is the maximal ideal space of $A$, then the following assertions are true:

**a:** There exists a unique resolution of the identity $E$ on the Borel subsets of $K$ that satisfies

$$T = \int_K \hat{T} dE$$

for every $T \in A$, where $\hat{T}$ is the Gelfand transform of $T$.

**b:** $E(\omega) \neq 0$ for every nonempty open set $\omega \subset K$.

**c:** An operator $S \in B(H)$ commutes with every $T \in A$ if and only if $S$ commutes with every projection $E(\omega)$.

Next, as promised, the spectral theorem for a (bounded) normal operator:

**Theorem 12.4.** If $T \in B(H)$ and $T$ is normal, then there exists a unique resolution of the identity $E$ on the Borel sets of $\sigma(T)$ which satisfies

$$T = \int_{\sigma(T)} \lambda dE(\lambda).$$

Furthermore, every projection $E(\omega)$ commutes with every $S \in B(H)$ which commutes with $T$.

From the last sentence it follows that the von Neumann algebra generated by $T$, $T^*$ and $1$ is generated by the projections $E(\omega)$.

We now move on to the spectral theorem for unbounded normal operators. Again, we avoid including proofs, as the use of the theorem is more important in these notes than an understanding of the proof. Recall, first, that an operator $T : H \to H$ is a normal operator on $H$ if $T$ is a closed, densely defined operator such that $T^*T = TT^*$.

**Lemma 12.5.** Let $f : \mathcal{M} \to \mathbb{C}$ be measurable. Put

$$D_f = \{\xi \in H : \int \Omega |f|^2 dE_{\xi,\xi} < \infty\}.$$
Then $D_f$ is a dense subspace of $\mathcal{H}$. If $\xi \in \mathcal{H}$ and $\eta \in \mathcal{H}$, then
\[
\int_{\Omega} |f| d|E_{\xi,\eta}| \leq ||\eta|| \left\{ \int_{\Omega} |f|^2 E_{\xi,\xi} \right\}^{1/2}.
\]
If $f$ is bounded and $\varpi = \Psi(f)\zeta$, then
\[
dE_{\xi,\varpi} = \overline{f}dE_{\xi,\zeta} \quad (\xi \in \mathcal{H}, \zeta \in \mathcal{H}).
\]

**Theorem 12.6.** Let $E$ be a resolution of the identity on a set $\Omega$.

**a:** To every measurable $f : \Omega \to \mathbb{C}$ corresponds a densely defined closed operator $\Psi(f)$ in $\mathcal{H}$, with domain $D(\Psi(f)) = D_f$, which is characterized by
\[
\langle \Psi(f)\xi, \eta \rangle = \int_{\Omega} f dE_{\xi,\eta} \quad (\xi \in D_f, \eta \in \mathcal{H})
\]
and which satisfies
\[
||\Psi(f)\xi||^2 = \int_{\Omega} |f|^2 dE_{\xi,\xi} \quad (\xi \in D_f).
\]

**b:** If $f$ and $g$ are measurable, then
\[
\Psi(f)\Psi(g) \subset \Psi(fg)
\]
\[
D(\Psi(f)\Psi(g)) = D_g \cap D_{fg}
\]
hence $\Psi(f)\Psi(g) = \Psi(fg)$ if and only if $D_{fg} \subset D_g$.

**c:** For every measurable $f : \Omega \to \mathbb{C}$,
\[
\Psi(f)^* = \Psi(\overline{f})
\]
and
\[
\Psi(f)\Psi(f)^* = \Psi(|f|^2) = \Psi(f)^*\Psi(f).
\]
Furthermore if $f : \Omega \to \mathbb{C}$ is measurable, $D_f = \mathcal{H}$ if and only if $f \in L^\infty(E)$.

Now we state the spectral theorem for (possibly unbounded) normal operators.

**Theorem 12.7.** For every normal operator $T$ on $\mathcal{H}$ there is a unique resolution of the identity $E$ on the Borel sets of $\sigma(T)$ such that
\[
\langle T\xi, \eta \rangle = \int_{\sigma(T)} \lambda dE_{\xi,\eta}(\lambda) \quad (\xi \in D(T), \eta \in \mathcal{H}).
\]
Moreover, $E(\omega)S = SE(\omega)$ for every Borel set $\omega \subset \sigma(T)$ and for every $S \in B(\mathcal{H})$ that commutes with $T$, in the sense that $ST \subset TS$. (Thus the operator $T$ is affiliated with the abelian von Neumann algebra generated by the $E(\omega)$) We note that $E$ is concentrated on $\sigma(T)$, in the sense that $E(\sigma(T)) = 1$.

**Theorem 12.8.** If $T$ is a self-adjoint operator on $\mathcal{H}$, then $T \geq 0 \iff \langle T\xi, \xi \rangle \geq 0$ whenever $\xi \in D(T)$ if and only if $\sigma(T) \subset [0, \infty)$. Also, if $T \geq 0$ then there exists a unique self-adjoint $B \geq 0$ such that $B^2 = T$. 
13. Appendix B: Integrable Operators and Noncommutative $L^p$-Spaces

With the functional calculus from Appendix A at our disposal, we may discuss integrable operators and Noncommutative $L^p$-spaces. What we consider here can be found, with greater generality, on pages 383-384 of [DL].

Suppose that $N$ is a finite von Neumann algebra with trace $\tau$ and consider $L^2(N) = L^2(N, \tau)$. Given $x \in N$ define the operator $L_x$ on $L^2(N)$ having domain $N$:

$$L_x y \Omega = xy \Omega.$$

Since $y^* x^* xy \leq ||x||^2 y^* y$, the operator $L_x$ is bounded and the GNS representation $\pi : x \mapsto L_x$ is a faithful and normal representation of $N$ on $L^2(N)$. Let $\mathcal{U}(N)$ be the collection of closed, densely defined operators on $L^2(N)$ affiliated to $N$. These are precisely the operators of the form $L_\xi$ with $\xi \in L^2(N)$, where $L_\xi y \Omega := Jy^* J\xi$ for all $y \in N$ (CHECK THIS AGAIN). Letting $\mathcal{U}_{s.a.}(N)$ denote the self-adjoint operators affiliated with $N$, we have that every $x \in \mathcal{U}_{s.a.}(N)$ has an associated resolution of the identity $E^{(x)}$ prescribed by the spectral theorem. We then note that under these circumstances the map $\nu^{(x)} : U \mapsto \tau(E^{(x)}(U))$ defines a positive Borel measure on $\mathbb{R}$. By linearity, normality and the monotone convergence theorem:

$$\tau(x) = \int_0^{||x||} \lambda d\nu^{(x)}(\lambda), \quad x \in \pi(N)_+$$

and observe that

$$\tau(x) = \int_0^\infty \lambda d\nu^{(x)}(\lambda)$$

extends $\tau$ faithfully to a (semifinite tracial CHECK) weight (CHECK) on $\mathcal{U}_{s.a.}(N)_+$. Given an operator $x \in \mathcal{U}(N)$, the operator $|x| \in \mathcal{U}_{s.a.}(N)_+$ is available and we define

$$||x||_\infty = \inf\{\lambda > 0 : \nu^{(|x|)}(\lambda, \infty) = 0\}$$

and for $p \in [1, \infty)$

$$||x||_p = \tau(||x||^p)^{1/p}.$$  

The space $L^p(N, \tau) = \{x \in \mathcal{U}(N) : ||x||_p < \infty\}$. Note that $\mathcal{U}(N) = \{x \in \mathcal{U}(N) : \nu^{(|x|)}(\lambda, \infty) < \infty \text{ for some } \lambda < \infty\}$ (CHECK THIS). With this prescription, $L^\infty(N, \tau) = \pi(N)$ and with strong sense operations (take the closure of the operator obtained from performing the operation), $\mathcal{U}(N)$ becomes a topological $*$-algebra in Stinespring’s topology of convergence in measure (INCLUDE DEFINITION AT LEAST OF THIS, SEE THOM PAPER). With this structure, we may refer to $\mathcal{U}(N)$ as the algebra of $\tau$-measurable operators. We suppress the notation for strong sense operations, as context will dictate when they are to be employed.

Each $L^p(N)$ is a vector subspace of $\mathcal{U}(N)$ under strong sense sum and scalar product, and $|| \cdot ||_p$ is a complete norm under which the adjoint operation is an isometry [At least refer to the page and paragraph of Segal’s paper].

The following result summarizes the basic properties of noncommutative $L^p$-spaces. This appears as Proposition 1.1 of [DL]:

**Proposition 13.1.**
\textbf{i:} If \( x, y \in L^p(N)_+ \) and \( x \leq y \), then
\[ ||x||_p \leq ||y||_p. \]

\textbf{ii:} If \( x, y \in \mathcal{U}(N) \), \( p, q, r \in [1, \infty] \) and \( r^{-1} = p^{-1} + q^{-1} \), then
\[ ||xy||_r \leq ||x||_p ||y||_q. \]

\textbf{iii:} If \( p \in [1, \infty) \) and \( \varphi \in (L^p(N))^* \) then there is \( x_\varphi \in L^{p'}(N) \), where \( p' = 1 + (p - 1)^{-1} \), for which
\[ \varphi(y) = \tau(x_\varphi y), \quad ||x_\varphi||_{L^{p'}(N)} = ||\varphi||_{(L^p(N))^*}. \]

\textbf{iv:} If \( p \in [1, \infty], x \in L^p(N) \) and \( y \in L^{p'}(N) \) then
\[ \tau(xy) = \tau(yx) \]
moreover, if also \( x, y \geq 0 \) then \( \tau(xy) \geq 0. \)