EXPLICIT FORMULAS FOR MULTIVARIABLE EULER AND BERNOULLI NUMBERS

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Abstract. By directly considering Taylor coefficients, we employ a generalized Faà di Bruno formula for higher partial derivatives to obtain identities that include explicit formulas for multivariable analogues of the Euler and Bernoulli numbers.

1. Introduction

The purpose of this paper is to extend the results of [Ve] to the multivariable setting. Recall that the sequences of Bernoulli numbers $B_n$ and Euler numbers $E_n$ have exponential generating functions $\frac{x}{e^x-1}$ and $\text{sech}(x)$, respectively. In [Ve] the following identities are obtained:

$$B_n = \sum_{\pi \in \mathcal{P}_n} \frac{(-1)^m}{1 + m} \binom{m}{\lambda(\pi)} \binom{n}{\pi} = \sum_{\pi \in \mathcal{C}_n} \frac{(-1)^m}{1 + m} \binom{n}{\pi}$$

$$B_n = \sum_{1 \leq m \leq n} \frac{(-1)^m m!}{1 + m} S(n, m)$$

$$E_n = \sum_{\pi \in \mathcal{P}_n} \binom{m}{\lambda(\pi)} \binom{n}{\pi} = \sum_{\pi \in \mathcal{C}_n} \binom{n}{\pi}$$

$$E_n = \sum_{1 \leq m \leq n} (-1)^m m! S(n, m, \text{even})$$

$$1 = \sum_{1 \leq r \leq j} \frac{(-1)^r}{(2r)!} E_{2r} \sum_{\pi \in \mathcal{P}_{2j}, \text{odd parts}} \binom{2r}{\lambda(\pi)} \binom{2j}{\pi} \prod_{s=0}^{j} \left[ E_{2s} \right]^{\frac{1}{2s+1}} \forall j > 0$$

where $\mathcal{P}_n$ is the set of integer partitions of $n$, $\mathcal{C}_n$ is the set of all ordered partitions (i.e. compositions) of $n$, $S(n, m)$ is the Stirling number of the second kind, i.e. the number of ways of partitioning a set of $n$ elements into exactly $m$ nonempty subsets, and $S(n, m, \text{even})$ is the number of ways of partitioning a set of $n$ elements into exactly $m$ nonempty subsets each with even cardinality.

Let $h_B(x_1, \ldots, x_d) = \frac{x_1 + \ldots + x_d}{e^{x_1 + \ldots + x_d} - 1}$ and $h_B(x_1, \ldots, x_d) = \text{sech}(x_1 + \ldots + x_d)$ be functions from $\mathbb{R}^d$ into $\mathbb{R}$. For a multi-index $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{N}_0^d$ we consider the generalized Bernoulli number $B_{\nu}$ (resp. $E_{\nu}$) to be $\nu!$ times the $\nu$'th Taylor coefficient of $h_B$ (resp. $h_E$). These generalized Bernoulli numbers were recently introduced and studied in

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[DO] in connection with multivariable Lévy processes. In the present paper we prove the precise analogues of the identities above for these multivariable Bernoulli numbers by applying the multivariable Faá di Bruno formula found in [CS].

The point of view adopted in [Ve] is that thinking explicitly about Taylor coefficients yields tools with a lot of combinatorial leverage. The results of the present paper rely even more heavily on this point of view. For example, it would be interesting to have a combinatorial interpretation for the analogue of $S(n, m)$ that appears in our new formulas, but we obtain these formulas without such a combinatorial interpretation.

The authors thank David Vella for visiting us to speak about the results of [Ve] which inspired the current paper.

2. Notation and Background

In this section, we fix notation that parallels that used in [CS], but will in the end yield formulas looking like those in [Ve]. We also restate the results from [CS] in our notation. Below let $\mathbb{N}$ denote the set of natural numbers, $\mathbb{N}_0$ the set of nonnegative integers. We regard finite cartesian powers such as $\mathbb{N}_0^d$, $\mathbb{N}^d$ as sitting in the natural way in the real vector space $\mathbb{R}^d$ throughout. For $\nu = (\nu_1, \ldots, \nu_d) \in \mathbb{N}_0^d$ and $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we remind the reader of the standard multi-index notation found on page 504 of [CS]:

$$|\nu| = \sum_{i=1}^d \nu_i, \quad \nu! = \prod_{i=1}^d (\nu_i!), \quad D^\nu_x = \frac{\partial^{|\nu|}}{\partial x_1^{\nu_1} \partial x_2^{\nu_2} \cdots \partial x_d^{\nu_d}}, \text{ for } |\nu| > 0, \quad D^0_x = \text{identity operator}, \quad \pi^\nu = \prod_{i=1}^d x_i^{\nu_i}. $$

Note that $|\cdot|$ will also be used for the cardinality of sets and multisets, the latter of which will be indicated by set notation using double brackets $\{\cdot\}$. Also, for $w = (w_1, \ldots, w_d) \in \mathbb{N}_0^d$, write $w \leq \nu$ if $w_i \leq \nu_i$ for $i = 1, 2, \ldots, d$ and write $w \prec \nu$ if one of the following holds: $|w| < |\nu|$, $|w| = |\nu|$ and $w_1 < \nu_1$, or $|w| = |\nu|$ and $w_1 = \nu_1$, $w_2 = \nu_2$, $\ldots$, $w_k = \nu_k$ and $w_{k+1} < \nu_{k+1}$ for some $1 \leq k < d$. One readily checks that $\prec$ defines a total ordering on $\mathbb{N}_0^d$. A function $h \in C^\nu(\mathbf{z}^0)$ if $D^\nu_x h$ exists and is continuous in a neighborhood of $\mathbf{z}^0$ for all $\mathbf{w} \leq \nu$. A function $h \in C^\nu(\mathbf{z}^0)$ if $h \in C^\nu(\mathbf{z}^0)$ for all $\nu \in \mathbb{N}_0^d$ with $|\nu| \leq n$. If $f : \mathbb{R}^\mu \rightarrow \mathbb{R}$ is a function for which $D^\nu_x f(\mathbf{z}^0)$ exists, then we let $T_\nu(f; \mathbf{z}^0) = \frac{1}{\nu!} D^\nu_x f(\mathbf{z}^0)$.

Now let $g : \mathbb{R}^d \rightarrow \mathbb{R}^\mu$ and $f : \mathbb{R}^\mu \rightarrow \mathbb{R}$ be functions and $h : \mathbb{R}^d \rightarrow \mathbb{R}$ their composition, i.e. let $h(x_1, \ldots, x_d) = f(g^{(1)}(x_1, \ldots, x_d), \ldots, g^{(\mu)}(x_1, \ldots, x_d))$. Assume that $0 \neq \nu = (\nu_1, \ldots, \nu_d) \in \mathbb{N}_0^d$ and $\mathbf{x}^0 = (x_1^0, \ldots, x_d^0) \in \mathbb{R}^d$ are given, $g^{(1)}, \ldots, g^{(\mu)} \in C_\mathbf{d}(\mathbf{x}^0)$ and $f \in C^{\nu}(\mathbf{y}^0)$, where $\mathbf{y}^0 = (g^{(1)}(\mathbf{x}^0), \ldots, g^{(\mu)}(\mathbf{x}^0))$. Then, setting $h_\nu = D^\nu_x h(\mathbf{x}^0)$,
Theorem 2.1. Faà di Bruno formula that appears as the main result (Theorem 2.1) of [CS]:

\[ f_m = D_y^m f(y^0), \quad g_k^{(i)} = D_x^k g^{(i)}(x^0), \quad g_k = (g_k^{(1)}, ..., g_k^{(\mu)}) \]

we can state the multivariable Faà di Bruno formula as follows. In a partial analogy to the one-variable case, we may regard the elements of \( D(f^y_m, \nu, m) \) as their respective multiplicities. Let \( p_s(\nu, m) = \{(m_1, ..., m_s; p_1, ..., p_s) : |m_i| > 0, 0 < p_1 < ... < p_s, \sum_{i=1}^s m_i = m \} \) and \( \sum_{i=1}^s m_i |p_i = \nu \} \).

In the above, the vector \( m = (r_1, ..., r_\mu) \in \mathbb{N}_0^\mu \) and we always set \( 0^0 = 1 \).

The elements of \( p_s(\nu, m) \) are called vector partitions of \( \nu \) of size \( s \) and total multiplicity \( m \). Let \( p(\nu, m) = \bigcup_{s=1}^\nu p_s(\nu, m) \) and \( p(\nu) = \bigcup_{s=1}^\nu p_s(\nu) \) in what follows. In a partial analogy to the one-variable case, we may regard the \( p_i \)'s as the parts of \( \nu \) and the \( |m_i| \)'s as their respective multiplicities.

**Corollary 2.2.**

\[ T_\nu(h; x^0) = \sum_{1 \leq |m| \leq |\nu|} T_m(f; y^0) \sum_{\pi \in p(\nu, m)} \left( \frac{m}{\lambda(\pi)} \right) \prod_{j=1}^{s} \prod_{k=1}^{\mu} T_{p_j}(g^{(k)}; x^0)^{|m_j|}_k \]

where \( \lambda(\pi) = \prod_{j=1}^{s} (m_j!) \) and \( \left( \frac{m}{\lambda(\pi)} \right) = \frac{m!}{\lambda(\pi)} \). Equivalently,

\[ T_\nu(h; x^0) = \sum_{\pi \in p(\nu)} \left( \frac{m}{\lambda(\pi)} \right) T_m(f; y^0) \prod_{j=1}^{s} \prod_{k=1}^{\mu} T_{p_j}(g^{(k)}; x^0)^{|m_j|}_k \]

**Proof.** This follows directly from (1), since

\[
\frac{h_\nu}{\nu!} = \sum_{1 \leq |m| \leq |\nu|} \left( \frac{f_m}{m!} \right) m! \sum_{s=1}^{|\nu|} \sum_{\pi \in p(\nu, m)} \prod_{j=1}^{s} \frac{1}{(m_j!)} \prod_{j=1}^{s} \frac{[g_{p_j}]^{|m_j|}}{[p_j!]^{|m_j|}} \\
= \sum_{1 \leq |m| \leq |\nu|} \left( \frac{f_m}{m!} \right) m! \sum_{s=1}^{|\nu|} \sum_{\pi \in p(\nu, m)} \frac{m!}{\prod (m_j!)} \prod_{j=1}^{s} \frac{\prod_{k=1}^{\mu} [g_{p_j}]^{|m_j|}}{\prod_{j=1}^{s} p_j!} \\
= \sum_{1 \leq |m| \leq |\nu|} \left( \frac{f_m}{m!} \right) m! \prod_{j=1}^{s} \frac{\prod_{k=1}^{\mu} [g_{p_j}]^{|m_j|}}{p_j!} \\
= \sum_{\pi \in p(\nu)} \left( \frac{m}{\lambda(\pi)} \right) \left( \frac{f_m}{m!} \right) \prod_{j=1}^{s} \frac{\prod_{k=1}^{\mu} [g_{p_j}]^{|m_j|}}{p_j!} \\
= \sum_{\pi \in p(\nu)} \left( \frac{m}{\lambda(\pi)} \right) T_m(f; y^0) \prod_{j=1}^{s} \prod_{k=1}^{\mu} T_{p_j}(g^{(k)}; x^0)^{|m_j|}_k 
\]
We’ll also make use of Theorem 3.4 of [CS]:

**Theorem 2.3.**

\[ h_\nu = \nu! \sum_{1 \leq |m|} \frac{f_m}{m!} \sum_{\Pi \in s(\nu, m)} \prod_{i=1}^{\mu} \prod_{j=1}^{r_i} \frac{[g_{(i)}^{(j)}]}{[p_j^{(i)}]!}, \]

where

\[ s(\nu, m) = \{ (p_1^{(1)}, \ldots, p_i^{(1)}, \ldots; p_1^{(\mu)}, \ldots, p_i^{(\mu)}) : \rho_j^{(i)} \in \mathbb{N}_0^{\mu} \text{ and } \sum_{i=1}^{\mu} \sum_{j=1}^{r_i} \rho_j^{(i)} = \nu \}. \]

In what follows, we’ll need the set

\[ s^+(\nu, m) = \{ (p_1^{(1)}, \ldots, p_i^{(1)}, \ldots; p_1^{(\mu)}, \ldots, p_i^{(\mu)}) \in s(\nu, m) : \rho_j^{(i)} \neq 0 \text{ if } i \in \{1, \ldots, \mu\}, j \in \{1, \ldots, r_i\} \}. \]

Let \( s^+(\nu) = \bigcup_{1 \leq |m|} s^+(\nu, m) \) in what follows.

**Proposition 2.4.**

\[ (4) \quad T_\nu(h; x^0) = \sum_{s^+(\nu)} T_m(f; y^0) \prod_{i=1}^{\mu} \prod_{j=1}^{r_i} T_{p_j^{(i)}} (g^{(i)}; x^0) \]

*Proof.* Directly substituting the right hand side of formula 3.8 of Corollary 3.3 of [CS] into formula 3.3 of [CS], we can achieve the following:

\[
\begin{align*}
\frac{h_\nu}{\nu!} &= \sum_{1 \leq |m|} \frac{f_m}{m!} \sum_{\Pi \in s(\nu, m)} \prod_{i=1}^{\mu} \prod_{j=1}^{r_i} \frac{[g_{(i)}^{(j)}]}{[p_j^{(i)}]!} = \sum_{1 \leq |m| \leq |\nu|} \frac{f_m}{m!} \sum_{\pi \in p(\nu, m)} \prod_{j=1}^{\pi} \frac{[g_{p_j}]^{m_j}}{(m_j)![p_j!]^{m_j}} \\
&= \sum_{1 \leq |m| \leq |\nu|} \frac{f_m m!}{m!} \sum_{\pi \in p(\nu, m)} \prod_{j=1}^{\pi} \frac{[g_{p_j}]^{m_j}}{(m_j)![p_j!]^{m_j}} \\
&= \sum_{1 \leq |m| \leq |\nu|} \frac{f_m}{m!} \sum_{\pi \in s^+(\nu, m)} \prod_{i=1}^{\mu} \prod_{j=1}^{r_i} \frac{[g_{(i)}^{(j)}]}{[p_j^{(i)}]!} \\
&= \sum_{\pi \in s^+(\nu)} \frac{f_m}{m!} \prod_{i=1}^{\mu} \prod_{j=1}^{r_i} \frac{[g_{(i)}^{(j)}]}{[p_j^{(i)}]!}.
\end{align*}
\]

\[ \square \]
3. Multivariable Bernoulli and Euler Number Identities

Recall from the introduction that if \( h_B(x_1, ..., x_d) = \frac{x_1 + ... + x_d}{e^{x_1 + ... + x_d} - 1} \) and \( h_B(x_1, ..., x_d) = \text{sech}(x_1 + ... + x_d) \) then the \( \nu^\text{th} \) generalized Bernoulli and Euler number is \( B_\nu = \nu! T_\nu(h_B; 0) \) and \( E_\nu = \nu! T_\nu(h_E; 0) \), respectively.

Recall the multivariable Stirling number of the second kind

\[
S(\nu, m) = \sum_{p(\nu, m)} \nu! \prod_{j=1}^{[\nu]} \frac{1}{m_j! (p_j!)^{m_j}} = \sum_{p(\nu, m)} \frac{\nu!}{\lambda(\pi)! \pi!}
\]

introduced on page 516 of [CS]. Define

\[
p(\nu, m, \text{even}) := \{ (m_1, ..., m_s; p_1, ..., p_s) \in p(\nu, m) : |p_j| \text{ even } \forall j \in \{1, ..., s\} \},
\]

\[
s^+(\nu, m, \text{even}) := \{ (p_1^{(1)}, ..., p_s^{(1)}; ..., p_1^{(\mu)}, ..., p_s^{(\mu)}) \in s^+(\nu, m) : |p_j^{(i)}| \text{ even } \forall i \in \{1, ..., \mu\} \forall j \in \{1, ..., r_\mu\} \},
\]

We analogously define \( p(\nu, m, \text{odd}) \) and \( s^+(\nu, m, \text{odd}) \). Let

\[
p(\nu, \text{even}) = \bigcup_{1 \leq |m| \leq |\nu|} p(\nu, m, \text{even}),
\]

\[
s^+(\nu, \text{even}) = \bigcup_{1 \leq |m| \leq |\nu|} s^+(\nu, m, \text{even}),
\]

and respectively define \( p(\nu, \text{odd}) \) and \( s^+(\nu, \text{odd}) \). We call the \( p_i \) appearing in elements of \( p(\nu, \text{even}) \) (resp. \( p(\nu, \text{odd}) \)) even parts (resp. odd parts) of \( \nu \).

Furthermore, let

\[
S(\nu, m, \text{even}) := \sum_{p(\nu, m, \text{even})} \nu! \prod_{j=1}^{[\nu]} \frac{1}{m_j! (p_j!)^{m_j}},
\]

and similarly define \( S(\nu, m, \text{odd}) \).

In what follows, for \( \pi \in p(\nu) \) and \( \Pi \in s^+(\nu) \), define \((\nu^\pi) = \frac{\nu^!}{\pi^!} \) with \( \pi^! = \prod_{j=1}^{[\nu]} (p_j)!^{m_j} \) and \((\nu^\Pi) = \frac{\nu^!}{\Pi^!} \) with \( \Pi^! = \prod_{i=1}^{\mu} \prod_{j=1}^{r_i} (p_j^{(i)})! \). We are now ready to give some explicit identities for multivariable Bernoulli numbers.

**Theorem 3.1.** For all \( m \in \mathbb{N} \),

(a) \[ B_\nu = \sum_{\pi \in p(\nu)} (-1)^m \left( \frac{m}{1 + m} \right) (\nu^\pi) = \sum_{\Pi \in s^+(\nu)} (-1)^m \left( \frac{\nu}{1 + m} \right) (\Pi) \]

(b) \[ B_\nu = \sum_{1 \leq m \leq |\nu|} (-1)^m m! \left( \frac{1}{1 + m} \right) S(\nu, m). \]

**Proof.** Let \( g(x_1, ..., x_d) = e^{x_1 + ... + x_d} \) and \( f(y) = \frac{\ln(1+y)}{y} \). Let \( x^0 = 0 \in \mathbb{N}^d \). Then \( T_{p_j}(g; 0) = \frac{1}{p_j!} \) if \( p_j > 0 \), while \( T_m(f; y^0) = T_m(f; 0) = (-1)^m \frac{1}{1 + m} \). By Corollary 2.2

\[
T_\nu(h; 0) = \sum_{1 \leq m \leq |\nu|} (-1)^m \left( \frac{1}{1 + m} \sum_{\pi \in p(\nu, m)} \left( \frac{m}{\lambda(\pi)} \right) \prod_{j=1}^{s} \frac{1}{p_j!} \right) S^m_j.
\]
Since $B_\nu = \nu! T_\nu(f \circ g; 0)$, this yields part (a):

$$B_\nu = \nu! T_\nu(h; 0) = \sum_{1 \leq m \leq |\nu|} \frac{(-1)^m}{1 + m} \sum_{\pi \in \mathcal{P}(\nu, m)} \left( \frac{m}{\lambda(\pi)} \right) \nu! \pi!$$

$$= \sum_{\pi \in \mathcal{P}(\nu)} \frac{(-1)^m}{1 + m} \left( \frac{m}{\lambda(\pi)} \right) \nu! \pi! = \sum_{\Pi \in s^+(\nu)} \frac{(-1)^m}{1 + m} \left( \frac{\nu}{\Pi} \right),$$

by Proposition 2.4. Part (b) follows from part (a) because $m! S(\nu, m) = \sum_{\pi \in \mathcal{P}(\nu, m)} \left( \frac{m}{\lambda(\pi)} \right) \nu! \pi!$ by (5). Collecting together partitions of a fixed total multiplicity yields:

$$B_\nu = \sum_{1 \leq m \leq |\nu|} \frac{(-1)^m m!}{1 + m} S(\nu, m).$$

Our next theorem gives explicit identities for calculating multivariable Euler numbers.

**Theorem 3.2.** For all $m \in \mathbb{N},$

(a) \[ E_\nu = \sum_{\pi \in \mathcal{P}(\nu, \text{even})} (-1)^m \left( \frac{m}{\lambda(\pi)} \right) \left( \frac{\nu}{\pi} \right) = \sum_{\Pi \in s^+(\nu, \text{even})} (-1)^m \left( \frac{\nu}{\Pi} \right) \]

(b) \[ E_\nu = \sum_{1 \leq m \leq |\nu|} (-1)^m m! S(\nu, m, \text{even}). \]

**Proof.** Let $g(x_1, ..., x_d) = \cosh(x_1, ..., x_d)$ and $f(y) = \frac{1}{y}$. Let $x^0 = 0 \in \mathbb{N}^d$. Then $T_{p_j}(g; 0) = \frac{1}{p_j!}$ for even parts and $T_{p_j}(g; 0) = 0$ for odd parts, while $T_m(f; y^0) = T_m(f; 1) = (-1)^m$. By Corollary 2.2

$$T_\nu(h; 0) = \sum_{1 \leq m \leq |\nu|} (-1)^m \sum_{\pi \in \mathcal{P}(\nu, m)} \left( \frac{m}{\lambda(\pi)} \right) \prod_{j=1}^{s} [T_{p_j}(g; 0)]^{m_j},$$

but if any of the parts of $\pi$ are odd, the product vanishes. Thus, the sum becomes over partitions of only even parts:

$$T_\nu(h; 0) = \sum_{1 \leq m \leq |\nu|} (-1)^m \sum_{\pi \in \mathcal{P}(\nu, m, \text{even})} \left( \frac{m}{\lambda(\pi)} \right) \prod_{j=1}^{s} \left( \frac{1}{p_j!} \right)^{m_j}.$$

Since $E_\nu = \nu! T_\nu(h; 0)$, this yields part (a):

$$E_\nu = \nu! T_\nu(h; 0) = \sum_{1 \leq m \leq |\nu|} (-1)^m \sum_{\pi \in \mathcal{P}(\nu, m, \text{even})} \left( \frac{m}{\lambda(\pi)} \right) \nu! \pi!$$

$$= \sum_{\pi \in \mathcal{P}(\nu, \text{even})} (-1)^m \left( \frac{m}{\lambda(\pi)} \right) \left( \frac{\nu}{\pi} \right) = \sum_{\Pi \in s^+(\nu, \text{even})} (-1)^m \left( \frac{\nu}{\Pi} \right),$$

\qed
by Proposition 2.4. Part (b) follows from part (a) because \( m!S(\mathbf{\nu}, m) = \sum_{\pi \in p(\mathbf{\nu}, m)} \binom{m}{\lambda(\pi)} \binom{\nu}{\pi} \) by (5). Collecting together partitions of a fixed total multiplicity yields:

\[
E_\nu = \sum_{1 \leq m \leq |\nu|} (-1)^m m! S(\mathbf{\nu}, m, \text{ even}).
\]

\[\square\]

**Theorem 3.3.** For all \( m \in \mathbb{N} \),

\[
1 = \sum_{1 \leq r \leq |\nu|} \frac{(-1)^r}{(2r)!} E_{2r} \sum_{\pi \in p(\mathbf{\nu}, 2r, \text{ odd})} \binom{2r}{\lambda(\pi)} \binom{\nu}{\pi} \prod_{j=1}^{s} [E_{p_j}]^{m_j}.
\]

**Proof.** Let \( g(x_1, \ldots, x_d) = 2\tan^{-1}(e^{x_1 + \ldots + x_d} - \frac{\pi}{2}) \) be the multivariable analogue of the gudermannian function and \( f(y) = \sec(y) \). Let \( \mathbf{x}^0 = \mathbf{0} \). Notice that \( h(x_1, \ldots, x_d) = \sec(g(x_1, \ldots, x_d)) = \cosh(x_1 + \ldots + x_d) \). Then \( T_{\nu}(h; \mathbf{x}^0) = T_{\nu}(h; \mathbf{0}) = \frac{1}{2} \) when \(|\nu|\) is even and \( T_{\nu}(h; \mathbf{0}) = 0 \) otherwise, while \( T_m(f; \mathbf{y}^0) = T_m(f; \mathbf{0}) = \frac{(-1)^{|\mathbf{y}|}}{(2m)!} E_m \) when \( m \) is even and \( T_m(f; \mathbf{0}) = 0 \) when \( m \) is odd. Letting \(|\nu| = 2a \) and \( m = 2r \), we substitute \( T_{2r}(f; \mathbf{0}) = \frac{(-1)^r}{(2r)!} E_{2r} \) into (2) of Corollary 2.2 to yield:

\[
\frac{1}{\nu!} = \sum_{1 \leq 2r \leq |\nu|} \frac{(-1)^r}{(2r)!} E_{2r} \sum_{\pi \in p(\mathbf{\nu}, 2r)} \binom{2r}{\lambda(\pi)} \prod_{j=1}^{s} [T_{p_j}(g; \mathbf{x}^0)]^{m_j}.
\]

From the basic properties of the gudermannian function,

\[
g(x_1, \ldots, x_d) = \int_{0}^{x_1} \sech(x_1, \ldots, x_d) dx_1
\]

\[
= \sum_{j_1, \ldots, j_d = 0}^{\infty} \frac{E(j_1, \ldots, j_d)}{j_1! \ldots j_d!} \int_{0}^{x_1} x_1^{j_1} \ldots x_d^{j_d} dx_i
\]

\[
= \sum_{j_1, \ldots, j_d = 0}^{\infty} \frac{E(j_1, \ldots, j_d)}{j_1! \ldots (j_1 + 1)! \ldots j_d!} x_1^{j_1} \ldots x_d^{j_d}.
\]

Thus, \( T_{(j_1, \ldots, j_1+1, \ldots, j_d)}(g; \mathbf{x}^0) = T_{(j_1, \ldots, j_1+1, \ldots, j_d)}(g; \mathbf{0}) = \frac{E(j_1, \ldots, j_d)}{j_1! \ldots (j_1 + 1)! \ldots j_d!} \). It follows that \( T_{(j_1, \ldots, j_1+1, \ldots, j_d)}(g; \mathbf{x}^0) = 0 \) unless \(|(j_1, \ldots, j_1 + 1, \ldots, j_d)|\) is odd, because formula (a) of 3.2, implies that either \( E(j_1, \ldots, j_d) = 0 \) or it is possible to write \((j_1, \ldots, j_d)\) as the sum of only even parts. It follows that

\[
1 = \sum_{1 \leq m \leq |\nu|} \frac{(-1)^r}{(2r)!} E_{2r} \sum_{\pi \in p(\mathbf{\nu}, 2r, \text{ odd})} \binom{2r}{\lambda(\pi)} \binom{\nu}{\pi} \prod_{j=1}^{s} [E_{p_j}]^{m_j}.
\]

\[\square\]
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