A NOTE ON NON-RESIDUALLY SOLVABLE HYPERLINEAR
ONE-RELATOR GROUPS

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Abstract. We prove that various non-residually finite, non-residually solvable
groups of the form \( \langle a, b | r^w = r^2 \rangle \) are sofic.

This paper concerns the sofic property discussed in the survey [Pest08]. Particularly, we address Question 4.10 in this paper: the problem of Nate Brown asking whether or not every one relator group is sofic. In [Ban10], the first author proves that the example in [Baum69] of a non-residually finite non-residually solvable one relator group is a sofic group. The purpose of this paper is to exhibit more such examples in the following large class of non-residually solvable one-relator groups introduced in [BaMiTro07]. Let \( F_2 = \langle a, b \rangle \) denote the free group on two generators. Let \( r, w \in F_2 \) be two elements that do not commute. In [BaMiTro07], the authors show that the group

\[
\Gamma_{r,w} = \langle a, b | r^w = r^2 \rangle = \langle a, b | r = [r, (r^{-1})^w] \rangle
\]

has the same finite quotients as the group

\[
\langle a, b | r \rangle
\]

and is therefore not residually finite. We point out that none of the groups \( \Gamma_{r,w} \) are residually solvable, since \( r = [r, (r^{-1})^w] \) lies in every derived subgroup of \( \Gamma_{r,w} \). In [Ban10], it is shown that the group \( \Gamma_{ab,a} \) is sofic. The proof in [Ban10] uses the recent result, Corollary 3.4 of [DykCol10], that HNN extensions of sofic groups over amenable subgroups remain sofic. The proof in [Ban10] uses the fact that \( \Gamma_{ab,a} \) is an HNN extension of an amenable one-relator group. We shall extend this result to certain other of the groups \( \Gamma_{r,w} \). If \( r \) and \( w \) generate \( F_2 \), then \( \Gamma_{r,w} \) embeds naturally as a subgroup of \( \Gamma_{ab,a} \), and since the sofic property passes to subgroups, \( \Gamma_{r,w} \) is sofic. The first result of this short note is that there exist \( r, w \) that do not generate \( F_2 \), yet the group \( \Gamma_{r,w} \) is sofic. More precisely, we prove:

**Theorem 0.1.** The group \( \Gamma_{ab^{-1}ab} \) is sofic.

**Proof.** Since \( \Gamma_{ab^{-1}ab} = \langle a, b | (bab^{-1})^{-2}a^{-1}(bab^{-1})^{-1}a(bab^{-1})a^{-1}(bab^{-1})a \rangle \), following McCool and Schupp as in [McCSch73], we let \( a_0 = a \) and \( a_{-1} = bab^{-1} \) and realize \( \Gamma_{ab^{-1}ab} \) as the HNN extension

\[
\langle a_0, a_{-1}, t | (a_{-1})^{-2}a_0^{-1}(a_{-1})^{-1}a_0a_{-1}a_0^{-1}a_{-1}a_0, t^{-1}a_{-1}t = a_0 \rangle
\]

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of the one-relator group $H_1 = \langle a_0, a_{-1} | a_0(a_{-1})^{-2}a_0^{-1}(a_{-1})^{-1}a_0a_{-1}a_0^{-1}a_{-1} \rangle$, where by the Freiheitsatz $\langle a_{-1} \rangle$ and $\langle a_0 \rangle$ are copies of $\mathbb{Z}$ which in the HNN extension we identify by identifying $a_{-1}$ and $a_0$. Letting $b_1 = a_0a_{-1}a_0^{-1}$ and $b_0 = a_{-1}$ we may identify $H_1$ as the HNN extension
\[
\left\langle \langle b_0, b_1, s \rangle b_1^{-2}b_0^{-1}b_1b_0, \ s^{-1}b_1s = b_0 \right\rangle
\]
of the one relator group $H_2 = \langle b_0, b_1, s \rangle b_1^{-2}b_0^{-1}b_1b_0$, where we identify the two copies $\langle b_0 \rangle$ and $\langle b_1 \rangle$ of $\mathbb{Z}$ as above. By [CeGrig97], the group $H_2$ is amenable, and hence by the argument in [Ban10] the group $H_1$ is sofic. Since $\Gamma_{a,b^{-1}ab}$ is an HNN extension of a sofic group with respect to identified copies of the amenable group $\mathbb{Z}$, it follows that $\Gamma_{a,b^{-1}ab}$ is sofic.

Note that in the above proof, we use in an essential way that the identified subgroups are amenable and therefore invoke the full hypotheses of Corollary 3.4 of [DykCol10], whereas in [Ban10] the group $\Gamma_{ab,a}$ is an HNN extension of an amenable group and so any pair of identified subgroups would work. We next illustrate that there are groups of the form $\Gamma_{r,w}$ that do not in an obvious way fall to the method of [Ban10].

**Theorem 0.2.** The group $\Gamma_{a,b^2} = \langle a, b | a = [a, (a^{-1})^{b^2}] \rangle$ is isomorphic to $(G * \mathbb{Z}) *_{p_2} G$, where $G$ is a one-relator amenable group.

**Proof.** Since $\Gamma_{a,b^2} = \langle a, b | a^{-2}(b^2ab^{-2})a(b^2ab^{-2})^{-1} \rangle$, then letting $a_0 = a$ and $a_{-2} = b^2ab^{-2}$ we have that $\Gamma_{a,b^2}$ is isomorphic to the HNN extension
\[
\langle a_0, a_{-1}, a_{-2}, t | a_0^{-2}a_{-2}a_0(a_{-2})^{-1}, t^{-1}a_{-2}t = a_{-1}, t^{-1}a_{-1}t = a_0 \rangle
\]
of the one relator group $\langle a_0, a_{-1}, a_{-2} | a_0^{-2}a_{-2}a_0(a_{-2})^{-1} \rangle$ with the isomorphism from the free subgroup $\langle a_{-2}, a_{-1} \rangle$ with $\langle a_{-1}, a_0 \rangle$ extending the set map sending $a_{-2}$ to $a_{-1}$ and $a_{-1}$ to $a_0$. But the relator $a_0^{-2}a_{-2}a_0(a_{-2})^{-1}$ does not involve $a_{-1}$, hence $\langle a_0, a_{-1}, a_{-2} | a_0^{-2}a_{-2}a_0(a_{-2})^{-1} \rangle = \langle a_{-1} \rangle * \langle a_0, a_{-2} | a_0^{-2}a_{-2}a_0(a_{-2})^{-1} \rangle$. \hfill \Box

**References**


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