A Non-Residually Solvable Hyperlinear One-Relator Group

Jon P. Bannon

Abstract. In this short note, we prove that the group $\langle a, b | a = [a, a^b] \rangle$ is hyperlinear. Unlike the non residually finite Baumslag-Solitar groups, this group is not residually solvable.

1. Introduction

Let $\Gamma$ denote the one-relator group $\langle a, b | a^{-1}[a, a^b] \rangle$, where $a^b = bab^{-1}$ and $[a, a^b] = a^{-1}(a^b)^{-1}aa^b$. This group was introduced by G. Baumslag in [Baum69] as an example of a non-cyclic one-relator group with the property that all of its finite index quotients are cyclic. It follows that the group $\Gamma$ is not residually finite. Also, $\Gamma$ is not residually solvable, since $a$ lies in every one of the derived subgroups of $\Gamma$. A countable discrete group $G$ is hyperlinear if it can be embedded as a subgroup of the unitary group $U(\mathcal{R}^\omega)$ of an ultrapower $\mathcal{R}^\omega$ of the hyperfinite type $II_1$ factor $\mathcal{R}$ (cf. [Pest08]). Equivalently, $G$ is hyperlinear if the group von Neumann algebra $L(G)$ is embeddable into $\mathcal{R}^\omega$ (cf. [Pest08]). Proposition 4.14 of [Ueda09], establishes that every HNN extension of an $\mathcal{R}^\omega$-embeddable type $II_1$ factor over a hyperfinite von Neumann subalgebra is also $\mathcal{R}^\omega$-embeddable. We use this fact along with a new standard trick of McCool and Schupp for one-relator groups to prove that the group $\Gamma$ above is hyperlinear. The main interest in this example is that it is an example of a non residually solvable hyperlinear one-relator group, and thus our result sheds a little light on the question of Nate Brown asking whether every one-relator group is hyperlinear. In [Rad00], Radulescu proved that the non residually finite Baumslag-Solitar group $\langle a, b | ab^3a^{-1}b^{-2} \rangle$ is hyperlinear. Radulescu’s result is shown in [Pest08] to follow more simply from the fact that these Baumslag-Solitar groups are residually solvable, and hence sofic.

2. Main Result

Theorem 1. The group $\Gamma = \langle a, b | a^{-1}[a, a^b] \rangle$ is hyperlinear.

Proof. We apply a rewriting process due to McCool and Schupp (cf. [McCSch73]). Let $a_0 = a$ and $a_{-1} = bab^{-1}$. Note that the word $a^{-1}[a, a^b] = a^{-2}a^{-1}b^{-1}abab^{-1}$

1991 Mathematics Subject Classification. Primary 46L10; Secondary 20F65.

Key words and phrases. sofic group, hyperlinear group, one-relator group.
when rewritten in terms of $a_0$ and $a_{-1}$ becomes
\[ a_0^{-2}(a_{-1})^{-1}a_0(a_{-1}). \]
The group $H = \langle a_0, a_{-1} | a_0^{-2}(a_{-1})^{-1}a_0(a_{-1}) \rangle$ is amenable, essentially by the Tits alternative. Or, we may appeal to Theorem 1.2 of [CeGrig97], and note that $a_0^{-2}(a_{-1})^{-1}a_0(a_{-1})$ has exponent sum zero on $a_{-1}$, and can be obtained from $(a_{-1})a_0(a_{-1})^{-1}a_0^{-2}$ by inverting $a_{-1}$ and cyclically shifting, and hence $H$ is amenable. We then note that the group $\Gamma$ is isomorphic to the HNN extension
\[ H*_{\varphi} = \langle t, H | t^{-1}a_{-1}t = a_0 \rangle. \]
Now, consider the group von Neumann algebra $L(H*_{\varphi})$. By Corollary 3.5 of [Ueda05], this is isomorphic to a reduced HNN extension of the hyperfinite $II_1$ factor $\mathcal{R}$ over $L(\mathbb{Z})$. Therefore, by Proposition 4.14 of [Ueda09], $L(H*_{\varphi})$ is embeddable into an $\mathcal{R}^\omega$, and therefore $\Gamma$ is hyperlinear.

**Remark 1.** We wish to thank the referee for pointing out that recently it has been shown that any HNN extension of a sofic group over an amenable subgroup is sofic. Precisely, this is Corollary 3.4 of [DykCol10]. We may, in the above proof, replace Ueda’s result by this one and obtain that $\Gamma$ is, in fact, a sofic group.

References


[Rad00] F. Radulescu, *The von Neumann algebra of the non-residually finite Baumslag group $\langle a, b | a^{-1}b^3a^{-1}b^{-2} \rangle$ embeds into $\mathcal{R}^\omega$*, arXiv:math/0004172v3, 2000.
