**FINAL EXAM**

**December 14, 2012**

Name: ____________________________

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**INSTRUCTIONS**

1. This examination is closed book and closed notes. All your belongings except a pen or pencil and a calculator should be put away and your bookbag should be placed on the floor.

2. You will find one page of useful formulae on the last page of the exam.

3. Please show all your work in the space provided on each page. If you need more space, feel free to use the back side of each page.

4. **Academic dishonesty** (i.e., copying or cheating in any way) will result in a zero for the exam, and may cause you to fail the class.

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**IN ORDER TO RECEIVE MAXIMUM CREDIT, EACH SOLUTION SHOULD HAVE:**

1. A labeled picture or diagram, if appropriate.
2. A list of given variables.
3. A list of the unknown quantities (i.e., what you are being asked to find).
4. One or more free-body or force-interaction diagrams, as appropriate, with labeled 1D or 2D coordinate axes.
5. Algebraic expression for the net force along each dimension, as appropriate.
6. Algebraic expression for the conservation of energy or momentum equations, as appropriate.
7. An algebraic solution of the unknown variables in terms of the known variables.
8. A final numerical solution, including units, with a box around it.
9. An answer to additional questions posed in the problem, if any.
1. A tennis player hits a ball 2.0 m above the ground. The ball leaves his racquet with a speed of 20.0 m/s at an angle of 5° above the horizontal. The horizontal distance to the net is 7 m and the net is 1 m high. Does the ball clear the net? If so, by how much? If not, by how much does it miss?

**Solution:**

This is a 2D projectile motion problem. We are given the initial height of the ball above the ground, \( y_i = 2 \) m, its initial velocity, \( v_i = 20.0 \) m/s, and the angle above the horizontal with which the ball is hit, \( \theta = 5^\circ \). We are also given the horizontal distance to the net, \( x_f = 7 \) m, and the vertical height of the net, \( y_f = 1 \) m. We are asked whether the ball clears the net or not.

To solve the problem we first determine the initial velocity of the ball in the horizontal \((x)\) and vertical \((y)\) dimensions:

\[
\begin{align*}
v_{xi} &= v_i \cos \theta = (20.0 \text{ m/s}) \times \cos (5^\circ) = 19.9 \text{ m/s}, \\
v_{yi} &= v_i \sin \theta = (20.0 \text{ m/s}) \times \sin (5^\circ) = 1.74 \text{ m/s}.
\end{align*}
\]

Next, we use the horizontal kinematic equation of projectile motion to find the time it takes the ball to travel the distance to the net:

\[
\begin{align*}
x_f &= x_i + v_{xi} \Delta t \\
&= v_{xi} \Delta t \\
\Rightarrow \Delta t &= \frac{x_f}{v_{xi}} = \frac{7 \text{ m}}{19.9 \text{ m/s}} = 0.351 \text{ s}.
\end{align*}
\]

Finally, we use the vertical kinematic equation of projectile motion to find the vertical height of the ball as it reaches the net:

\[
\begin{align*}
y_f &= y_i + v_{yi} \Delta t - \frac{1}{2} g (\Delta t)^2, \\
&= (2.0 \text{ m}) + (1.74 \text{ m/s}) \times (0.351 \text{ s}) - \frac{1}{2} \times (9.8 \text{ m/s}^2) \times (0.351 \text{ s})^2 \\
&= 2.01 \text{ m}.
\end{align*}
\]

So yes, the ball clears the net by about one meter!
2. Sand moves without slipping at 6.0 m/s down a conveyor that is tilted at 15°. The sand enters a pipe 3.0 m below the end of the conveyor belt. What is the horizontal distance \( d \) between the conveyor belt and the pipe?

**Solution:**

This is a 2D projectile motion problem. We are given the initial velocity of the sand, \( v_i = 6.0 \text{ m/s} \), the angle of the conveyor belt, \( \theta = 15^\circ \), and the vertical distance to the top of the pipe, \( y_f = 3.0 \text{ m} \). We are asked to find the horizontal distance, \( x_f = d \), between the conveyor belt and the pipe.

To solve this problem we must recognize that once the sand leaves the conveyor belt it is a free-falling projectile. In other words, it maintains the same constant horizontal velocity it had when it left the belt, while its vertical velocity increases due to gravity.

First, we determine the velocity of the sand in the horizontal (\( x \)) and vertical (\( y \)) dimensions as it leaves the edge of the conveyor belt:

\[
\begin{align*}
v_{xi} &= v_i \cos \theta = (6.0 \text{ m/s}) \times \cos (15^\circ) = 5.80 \text{ m/s}, \\
v_{yi} &= -v_i \sin \theta = -(6.0 \text{ m/s}) \times \sin (15^\circ) = -1.55 \text{ m/s},
\end{align*}
\]

where note the negative sign on the \( v_{yi} \) velocity. Next, we use the vertical kinematic equation of projectile motion to find the time it takes the sand to fall to the top of the pipe (\( y_f = 0 \)):

\[
\begin{align*}
y_f &= y_i + v_{yi} \Delta t - \frac{1}{2} g (\Delta t)^2, \\
0 &= y_i + v_{yi} \Delta t - \frac{1}{2} g (\Delta t)^2
\end{align*}
\]

This is a quadratic equation for time of the form \( 0 = c + bz + az^2 \), with solution

\[
z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.\]

Solving for time, we get:

\[
\Delta t = \frac{-v_{yi} \pm \sqrt{v_{yi}^2 - 4y_i \times (-g/2)}}{2(-g/2)},
\]

\( \square \)
\[
\begin{align*}
    v_y &= \frac{-v_{yi} \pm \sqrt{v_{yi}^2 + 2y_ig}}{-g}, \quad (6) \\
    &= \frac{1.55 \text{ m/s} \pm \sqrt{(-1.55 \text{ m/s})^2 + 2 \times (9.8 \text{ m/s}^2) \times (3.0 \text{ m})}}{-9.8 \text{ m/s}^2}, \quad (7) \\
    &= \frac{1.55 \text{ m/s} \pm 7.82 \text{ m/s}}{-9.8 \text{ m/s}^2}, \quad (8) \\
    &= 0.640 \text{ s}. \quad (9)
\end{align*}
\]

Note that in the last line we had to adopt the “minus” solution to ensure that time was a positive number.

Finally, we can use the horizontal equation of motion to solve for the final distance, noting that the initial horizontal position \( x_i = 0 \):

\[
\begin{align*}
    x_f &= d = x_i + v_{xi}\Delta t, \quad (10) \\
    &= v_{xi}\Delta t, \quad (11) \\
    &= (5.80 \text{ m/s}) \times (0.640 \text{ s}), \quad (12) \\
    &= 3.7 \text{ m}. \quad (13)
\end{align*}
\]
3. A rock stuck in the tread of a 60.0 cm diameter bicycle wheel has a tangential speed of 3.00 m/s. When the brakes are applied, the rock’s tangential deceleration is 1.00 m/s².

(a) What are the magnitudes of the rock’s angular velocity and angular acceleration at \( t = 1.50 \) s?

(b) At what time is the magnitude of the rock’s acceleration equal to \( g \)?

**Solution:**

This is a non-uniform angular acceleration problem. We are given the size of the wheel, \( r = 30 \) cm = 0.30 m, the initial tangential speed of the rock, \( v_i = 3.00 \) m/s, and the tangential deceleration of the rock when the brakes are applied, \( a_T = 1.00 \) m/s².

(a) In order to determine the final angular velocity, \( \omega_f \) of the rock after \( t = 1.50 \) s, we first have to determine the initial angular velocity:

\[
\omega_i = \frac{v_i}{r} = \frac{3.00 \text{ m/s}}{0.30 \text{ m}} = 10 \text{ rad/s}. \tag{1}
\]

The angular acceleration is a constant and given by:

\[
\alpha = \frac{a_T}{r} = \frac{-1.00 \text{ m/s}^2}{0.30 \text{ m}} = -3.33 \text{ rad/s}^2. \tag{2}
\]

Finally, using the appropriate rotational kinematic equation and substituting, we obtain

\[
\omega_f = \omega_i + \alpha \Delta t \tag{3}
\]

\[
= 10 \text{ rad/s} - (3.33 \text{ rad/s}^2) \times (1.5 \text{ s}) \tag{4}
\]

\[
= 5.01 \text{ rad/s}. \tag{5}
\]
(b) In order to find the time at which the magnitude of the angular acceleration of the rock equals \(g\), we first need to find the linear velocity of the rock at that time. We want \(|a| = g = \sqrt{a_T^2 + a_r^2}\), where \(a_T\) is given in the problem and \(a_r = \frac{v^2}{r}\). Substituting and solving for \(v\) we get

\[
g = \sqrt{a_T^2 + \left(\frac{v^2}{r}\right)^2}
\]  
\quad \text{(6)}

\[
g^2 = a_T^2 + \frac{v^4}{r^2}
\]  
\quad \text{(7)}

\[
\Rightarrow v^4 = r^2(g^2 - a_T^2)
\]  
\quad \text{(8)}

\[
v = r^{1/2}(g^2 - a_T^2)^{1/4}
\]  
\quad \text{(9)}

\[
= (0.30 \text{ m})^{1/2} \times [(9.8 \text{ m/s}^2)^2 - (1.00 \text{ m/s}^2)^2]^{1/4}
\]  
\quad \text{(10)}

\[
= 1.71 \text{ m/s}.
\]  
\quad \text{(11)}

To find the time at which the rock reaches this velocity, we use the 1D kinematic equation for velocity and solve for the time:

\[
v_f = v_i - a_T \Delta t
\]  
\quad \text{(12)}

\[
\Rightarrow \Delta t = \frac{v_i - v_f}{a_T}
\]  
\quad \text{(13)}

\[
= \frac{3.00 \text{ m/s} - 1.71 \text{ m/s}}{1.00 \text{ m/s}^2}
\]  
\quad \text{(14)}

\[
= 1.29 \text{ s}.
\]  
\quad \text{(15)}
4. You and your friend Peter are putting new shingles on a roof pitched at $25^\circ$. You’re sitting on the very top of the roof when Peter, who is at the edge of the roof directly below you, 5.0 m away, asks you for the box of nails. Rather than carry the 2.5 kg box of nails down to Peter, you decide to give the box a push and have it slide down to him. If the coefficient of kinetic friction between the box and the roof is 0.55, with what speed should you push the box to have it gently come to rest right at the edge of the roof?

**Solution:**

The pictorial representation and free-body diagram are shown below:

This is a 1D dynamics problem. The relevant forces are gravity, $F_G$, the normal force, $n$, and the kinetic friction force, $f_k$. Note that we do not include the initial force that was applied to the box of nails to get it moving, but we do include the fact that the box has some initial velocity.

The most natural coordinate system is one that is rotated by $25^\circ$ and therefore aligned with the roof. The interaction and free-body diagrams are shown above. The net force along the $x$- and $y$-axis is

\[
(F_{\text{net}})_x = \sum F_x = F_G \sin 25^\circ - f_k = ma \quad (1)
\]
\[
(F_{\text{net}})_y = \sum F_y = n - F_G \cos 25^\circ = 0 \quad (2)
\]
\[
\Rightarrow n = F_G \cos 25^\circ \quad (3)
\]

In the second line we used the fact that the shingles are not leaping off the roof to set the acceleration in the $y$-direction equal to zero. The magnitude of the force of gravity is $F_G = mg$, and the magnitude of the kinetic force of friction is

\[
f_k = \mu_k n \quad (4)
\]
\[
= \mu_k F_G \cos 25^\circ \quad (5)
\]
\[
= \mu_k mg \cos 25^\circ. \quad (6)
\]
Substituting equation (6) into equation (1) and solving for the acceleration $a$ we get

$$
mg \sin 25^\circ - \mu_k mg \cos 25^\circ = ma
$$

(7)

$$
\Rightarrow a = (\sin 25^\circ - \mu_k \cos 25^\circ)g
$$

(8)

$$
= (\sin 25^\circ - 0.55 \times \cos 25^\circ) \times 9.8 \text{ m/s}^2
$$

(9)

$$
= -0.743 \text{ m/s}^2
$$

(10)

where the minus sign indicates the acceleration is directed up the incline. To find the initial speed, $v_i$, necessary to have the box come to rest (i.e., $v_f = 0$) after $\Delta x = 5.0 \text{ m}$ can be found from the kinematic equation linking velocity and acceleration:

$$
v_f^2 = v_i^2 + 2a\Delta x
$$

(11)

$$
\Rightarrow v_i = \sqrt{-2a\Delta x}
$$

(12)

$$
= \sqrt{-2(-0.743 \text{ m/s}^2)(5.0 \text{ m})}
$$

(13)

$$
= 2.7 \text{ m/s}.
$$

(14)
5. The 1.0 kg block in the figure below is tied to the wall with a rope. It sits on top of the 2.0 kg block. The lower block is pulled to the right with a tension force of 20 N. The coefficient of kinetic friction at both the lower and upper surfaces of the 2.0 kg block is $\mu_k = 0.40$.

(a) What is the tension in the rope holding the 1.0 kg block to the wall?

(b) What is the acceleration of the 2.0 kg block?

Solution:
The free-body diagram for the problem is shown below:

To solve this problem we need to use both Newton’s third and second laws. The separate free-body diagrams for the two blocks show that there are two action/reaction pairs. Notice how the top block (block 1) both pushes down on the bottom block
(block 2) with force \( \vec{n}'_1 \), and exerts a retarding friction force \( \vec{f}_{2,\text{top}} \) on the top surface of block 2.

(a) Block 1 is in static equilibrium \( (a_1 = 0 \text{ m/s}^2) \), but block 2 is accelerating to the right. Newton’s second law for block 1 is

\[
(F_{\text{net on 1}})_x = f_1 - T_{\text{rope}} = 0 \Rightarrow T_{\text{rope}} = f_1
\]

\[
(F_{\text{net on 1}})_y = n_1 - m_1g = 0 \Rightarrow n_1 = m_1g.
\]

Although block 1 is stationary, there is a kinetic force of friction because there is motion between block 1 and block 2. The friction model means

\[
f_1 = \mu_k n_1 = \mu_k m_1g.
\]

Substituting this result into equation (1) we get the tension of the rope:

\[
T_{\text{rope}} = f_1 = \mu_k m_1g
\]

\[
= (0.40) \times (1.0 \text{ kg}) \times (9.8 \text{ m/s}^2)
\]

\[
= 3.9 \text{ N}.
\]

(b) Newton’s second law for block 2 is

\[
a_x \equiv a = \frac{(F_{\text{net on 2}})_x}{m_2} = \frac{T_{\text{pull}} - f_{2,\text{top}} - f_{2,\text{bot}}}{m_2}
\]

\[
a_y = \frac{(F_{\text{net on 2}})_y}{m_2} = \frac{n_2 - n'_1 - m_2g}{m_2} = 0
\]

Forces \( n_1 \) and \( n'_1 \) are an action/reaction pair, so \( n'_1 = n_1 = m_1g \). Substituting into equation (8) gives

\[
n_2 = (m_1 + m_2)g.
\]

This result is not surprising because the combined weight of both objects presses down on the surface. The kinetic friction on the bottom surface of block 2 is then

\[
f_{2,\text{bot}} = \mu_k n_2 = \mu_k (m_1 + m_2)g.
\]

Next, we recognize that the forces \( \vec{f}_1 \) and \( \vec{f}_{2,\text{top}} \) are an action/reaction pair, so

\[
f_{2,\text{top}} = f_1 = \mu_k m_1g.
\]

Finally inserting these friction results into equation (7) gives

\[
a = \frac{T_{\text{pull}} - \mu_k m_1g - \mu_k (m_1 + m_2)g}{m_2}
\]

\[
= \frac{20 \text{ N} - (0.40)(1.0 \text{ kg})(9.8 \text{ m/s}^2) - (0.40)(1.0 \text{ kg} + 2.0 \text{ kg})(9.8 \text{ m/s}^2)}{2.0 \text{ kg}}
\]

\[
= 2.2 \text{ m/s}^2.
\]
6. The lower block in the figure below is pulled on by a rope with a tension force of 20 N. The coefficient of kinetic friction between the lower block and the surface is 0.30. The coefficient of kinetic friction between the lower block and the upper block is also 0.30. What is the acceleration of the 2.0 kg block?

![Diagram of two blocks](image)

**Solution:**

The pictorial representation and free-body diagrams are shown below:

![Free-body diagrams](image)

The blocks accelerate with the same magnitude but in opposite directions. Thus the acceleration constraint is \( a_2 = a = -a_1 \), where \( a \) will have a positive value because of how we have defined the \(+x\) direction. There are two real action/reaction pairs. The two tension forces will act as if they are action/reaction pairs because we are assuming a massless rope and a frictionless pulley.

Make sure you understand why the friction forces point in the directions shown in the free-body diagrams, especially force \( \vec{f}_1 \) exerted on the bottom block (block 2) by the top block (block 1).
We have quite a few pieces of information to include. First, Newton’s second law applied to block 1 gives

\[
(F_{\text{net on 1}})_x = f_1 - T_1 = \mu_k n_1 - T_1 = m_1 a_1 = -m_1 a \tag{1}
\]

\[
(F_{\text{net on 1}})_y = n_1 - m_1 g = 0 \tag{2}
\]

\[
\Rightarrow n_1 = m_1 g. \tag{3}
\]

For block 2 we have

\[
(F_{\text{net on 2}})_x = T_{\text{pull}} - f'_1 - f_2 - T_2 = T_{\text{pull}} - f'_1 - \mu_k n_2 - T_2 = m_2 a_2 = m_2 a \tag{4}
\]

\[
(F_{\text{net on 2}})_y = n_2 - n'_1 - m_2 g = 0 \tag{5}
\]

\[
\Rightarrow n_2 = n'_1 + m_2 g. \tag{6}
\]

Note that to simplify the two \(x\)-equations above we have already used the kinetic friction model. Next, from Newton’s third law we have three additional constraints:

\[
n'_1 = n_1 = m_1 g \tag{7}
\]

\[
f'_1 = f_1 = \mu_k n_1 = \mu_k m_1 g\tag{8}
\]

\[
T_1 = T_2 = T. \tag{9}
\]

Knowing \(n'_1\) we can now use the \(y\)-equation for block 2 to find \(n_2\). Substituting all these pieces into the two \(x\)-equations, we end up with two equations with two unknowns:

\[
\mu_k m_1 g - T = -m_1 a \tag{10}
\]

\[
T_{\text{pull}} - T - \mu_k m_1 g - \mu_k (m_1 + m_2)g = m_2 a. \tag{11}
\]

Subtracting the first equation from the second we get

\[
T_{\text{pull}} - T - \mu_k m_1 g - \mu_k (m_1 + m_2)g - \mu_k m_1 g + T = m_2 a + m_1 a \tag{12}
\]

\[
T_{\text{pull}} - 3\mu_k m_1 g - \mu_k m_2 g = (m_2 + m_1) a \tag{13}
\]

\[
T_{\text{pull}} - \mu_k (3m_1 + m_2) g = (m_2 + m_1) a. \tag{14}
\]

And finally solving for \(a\) we get

\[
\Rightarrow a = \frac{T_{\text{pull}} - \mu_k (3m_1 + m_2) g}{m_1 + m_2} \tag{15}
\]

\[
= \frac{20 \text{ N} - (0.30)(3 \times 1.0 \text{ kg} + 2.0 \text{ kg})(9.8 \text{ m/s}^2)}{1.0 \text{ kg} + 2.0 \text{ kg}} \tag{16}
\]

\[
= 1.8 \text{ m/s}^2. \tag{17}
\]
7. The 1.0 kg physics book in the figure below is connected by a string to a 500 g coffee cup. The book is given a push up the slope and released with a speed of 3.0 m/s. The coefficients of friction are \( \mu_s = 0.50 \) and \( \mu_k = 0.20 \).

(a) How far does the book slide?
(b) At the highest point, does the book stick to the slope, or does it slide back down?

Solution:
The pictorial representation and free-body diagrams are shown below:
To solve this problem we use the particle model for the book (B) and the coffee cup (C), the models of kinetic and static friction, and the constant-acceleration kinematic equations.

(a) To find the distance $x_1$ the book slides we must find its acceleration. Newton’s second law applied to the book gives

$$
\sum (F_{on\ B})_y = n_B - (F_G)_B \cos \theta = 0
$$

$$
\Rightarrow n_B = m_B g \cos \theta \tag{1}
$$

$$
\sum (F_{on\ B})_x = -T - f_k - (F_G)_B \sin \theta = -T - \mu_k n_B - m_B g \sin \theta = -T - \mu_k m_B g \cos \theta - m_B g \sin \theta = m_B a_B. \tag{2}
$$

Similarly, for the coffee cup we have

$$
\sum (F_{on\ C})_y = T - (F_G)_C = T - m_C g = m_C a_C. \tag{3}
$$

Equations (5) and (6) can be rewritten using the acceleration constrain $a_C = a_B = a$ as

$$
-T - \mu_k m_B g \cos \theta - m_B g \sin \theta = m_B a \tag{4}
$$

$$
T - m_C g = m_C a. \tag{5}
$$

Adding these two equations and solving for the acceleration gives

$$
-\mu_k m_B g \cos \theta - m_B g \sin \theta - m_C g = (m_B + m_C) a
\Rightarrow a = \frac{-m_B(\mu_k \cos \theta + \sin \theta) + m_C}{m_B + m_C} \times g
$$

$$
= \frac{-[(1.0 \text{ kg}) \times (0.20 \cos 20^\circ + \sin 20^\circ) + 0.5 \text{ kg}]}{1.0 \text{ kg} + 0.5 \text{ kg}} \times (9.8 \text{ m/s}^2) \tag{6}
$$

$$
= -6.73 \text{ m/s}^2. \tag{7}
$$

Finally, to solve for the distance $x_1$ we use the following kinematic equation with $v_{1x} = 0$, $v_{0x} = 3.0 \text{ m/s}$, and $x_0 = 0$:

$$
v_{1x}^2 = v_{0x}^2 + 2a(x_1 - x_0) \tag{8}
$$

$$
0 = v_{0x}^2 + 2a x_1 \tag{9}
$$

$$
\Rightarrow x_1 = \frac{-v_{0x}^2}{2a} \tag{10}
$$

$$
= \frac{-(3.0 \text{ m/s})^2}{2 \times (-6.73 \text{ m/s}^2)} \tag{11}
$$

$$
= 0.67 \text{ m}. \tag{12}
$$
(b) In order to figure out if the book sticks to the slope or slides back down we have to determine if the static friction force needed to keep the book in place, $f_s$ is larger or smaller than the maximum static friction force

$$\begin{align*}
(f_s)_{\text{max}} &= \mu_s n_B = \mu_s m_B g \cos \theta \\
&= (0.50) \times (9.8 \text{ m/s}^2) \times \cos 20^\circ \\
&= 4.60 \text{ N. (19)}
\end{align*}$$

When the cup is at rest, the string tension is $T = m_C g$. In this case, Newton’s first law for the book becomes

$$\begin{align*}
\sum (F_{\text{on } B})_x &= f_s - T - m_B g \sin \theta \\
&= f_s - m_C g - m_B g \sin \theta = 0 \\
\Rightarrow f_s &= (m_C + m_B \sin \theta) g \\
&= (0.5 \text{ kg} + 1.0 \text{ kg} \sin 20^\circ) \times (9.8 \text{ m/s}^2) \\
&= 8.25 \text{ N. (24)}
\end{align*}$$

Because $f_s > (f_s)_{\text{max}}$, the book slides back down.
8. A conical pendulum is formed by attaching a 500 g ball to a 1.0 m-long string, then allowing the mass to move in a horizontal circle of radius 20 cm. The figure shows that the string traces out the surface of a cone, hence the name.

(a) What is the tension in the string?
(b) What is the ball’s angular speed, in rpm?

Solution:

(a) What is the tension in the string?

The forces in the z-direction in the free body diagram are the component of the tension in the z-direction and the force of gravity. To find the z-component of the tension,
\[
\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{T_z}{T} \quad (1)
\]
\[
T_z = T \cos \theta \quad (2)
\]

Apply Newton’s second law in the \(z\)-direction

\[
\sum F_z = T \cos \theta - mg = 0 \quad (3)
\]
\[
T = \frac{mg}{\cos \theta} \quad (4)
\]

To calculate \(\cos \theta\),

\[
\cos \theta = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{\sqrt{L^2 - r^2}}{L} = \frac{\sqrt{(1\text{m})^2 - (0.2\text{m})^2}}{1.0\text{m}} = 0.98 \quad (5)
\]
\[
\Rightarrow \theta = 11.5^\circ \quad (6)
\]

Plugging this in for tension,

\[
T = \frac{mg}{\cos \theta} = \frac{(0.500 \text{ kg}) \times (9.80 \text{ m/s}^2)}{0.98} \quad (7)
\]
\[
T = 5.0 \text{ N} \quad (8)
\]

(b) What is the ball’s angular speed, in rpm?

Referring back to the tension triangle, the radial component of tension is

\[
\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{T_r}{T} \quad (9)
\]
\[
T_r = T \sin \theta \quad (10)
\]
Referring back to the triangle for the length of the pendulum,

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{r}{L}$$  \hspace{1cm} (11)

$$r = L \sin \theta$$  \hspace{1cm} (12)

Apply Newton’s second law in the \(r\)-direction

$$\sum F_r = T_r = T \sin \theta = m\omega^2 r$$  \hspace{1cm} (13)

$$\Rightarrow \omega = \sqrt{\frac{T \sin \theta}{mr}}$$  \hspace{1cm} (14)

$$= \sqrt{\frac{5.0 \text{ N} \times \sin 11.5^\circ}{0.500 \text{ kg} \times 0.2 \text{ m}}}$$  \hspace{1cm} (15)

$$= 3.16 \text{ rad/sec}$$  \hspace{1cm} (16)

$$= 3.16 \frac{\text{rad}}{\text{sec}} \times \left( \frac{1 \text{ rev}}{2\pi \text{ rad}} \right) \times \left( \frac{60 \text{ sec}}{1 \text{ min}} \right)$$  \hspace{1cm} (17)

$$= 30 \text{ rpm}$$  \hspace{1cm} (18)
9. A block of mass \( m \) slides down a frictionless track, then around the inside of a circular loop-the-loop of radius \( R \). From what minimum height \( h \) must the block start to make it around without falling off? Give your answer as a multiple of \( R \).

**Solution:**

This is a two-part problem. In the first part, we will find the critical velocity for the block to go over the top of the loop without falling off. Since there is no friction, the sum of the kinetic and gravitational potential energy is conserved during the block’s motion. We will use this conservation equation in the second part to find the minimum height the block must start from to make it around the loop.

Place the origin of the coordinate system directly below the block’s starting position on the frictionless track. From the free-body diagram we have

\[
-F_g - n = -mv_c^2/R.
\]  

(1)

For the block to just stay on the track, \( n = 0 \). Therefore, the critical velocity \( v_c \) is

\[
F_g = m\frac{v_c^2}{R} \]

(2)

\[
mg = m\frac{v_c^2}{R} \]

(3)

\[
\Rightarrow v_c = \sqrt{gR}
\]

(4)

We can now use conservation of mechanical energy to find the minimum height \( h \):

\[
K_f + U_f + K_i + U_i
\]

(5)

\[
\frac{1}{2}mv_f^2 + mgy_f = \frac{1}{2}mv_i^2 + mgy_i.
\]

(6)

Using \( v_f = v_c = \sqrt{gR} \), \( y_f = 2R \), \( v_i = 0 \), and \( y_i = h \), we get

\[
\frac{1}{2}gR + g(2R) = 0 + gh
\]

(7)

\[
\Rightarrow h = 2.5R.
\]

(8)
10. A pendulum is formed from a small ball of mass $m$ on a string of length $L$. As the figure shows, a peg is height $h = L/3$ above the pendulum’s lowest point. From what minimum angle $\theta$ must the pendulum be released in order for the ball to go over the top of the peg without the string going slack?

\[ \text{Solution:} \] This is a two part problem. First, we need to find the velocity for the ball to go over the peg without the string going slack. Then we need to find the potential energy to match that velocity.

(a) Find the velocity of the ball so that the string doesn’t go slack. We’ll set our origin on the peg. This means that once the string touches the peg, the ball will be moving in circular motion with a radius of $L/3$.

When the ball is let go, it moves below the peg and then moves in circular motion. When it reaches the top of the circle and is directly above the peg, we can use Newton’s second law and the fact that the ball is moving in circular motion.

\[
\sum F = ma = -m \frac{v^2}{R} \tag{1}
\]
\[
-T - F_g = -m \frac{v^2}{R} \tag{2}
\]
\[
(3)
\]
The critical speed is just as the tension equals zero. Factoring the negatives and the masses, and setting $F_g = mg$ and $T = 0$,

\[ 0 + mg = m\frac{v_c^2}{R} \]

\[ v_c = \sqrt{gR} \] (5)

Setting $R = L/3$,

\[ v_c = \sqrt{\frac{gL}{3}} \] (6)

(b) Find the angle to release the ball. Now that we know the velocity for the string to remain taut, we'll use the conservation of energy to find the height to release the ball.

\[ K_f + U_f = K_i + U_i \] (7)

\[ \frac{1}{2}mv_f^2 + mgy_f = \frac{1}{2}mv_i^2 + mgy_i \] (8)

\[ \frac{1}{2}v_c^2 + g y_f = 0 + gy_i \] (9)

Plugging in $y_f = L/3$ and $v_c^2 = gL/3$,

\[ \frac{gL}{6} + \frac{gL}{3} = gy_i \] (10)

\[ \frac{L}{6} + \frac{L}{3} = y_i \] (11)

\[ y_i = \frac{L}{2} \] (12)

This result tells us that we need to release the ball at a height of $L/2$ above the peg.

To find the angle, consider the third figure in the three-part figure above. We need to find the distance of the ball below the point where the string is attached. We can use the string length and result above.

\[ h = L - \frac{L}{2} - \frac{L}{3} = \frac{L}{6} \] (13)

To find the angle, we can use cosine,

\[ \cos \theta = \frac{L/6}{L} = \frac{1}{6} \] (14)

\[ \Rightarrow \theta = 80.4^\circ \] (15)
11. A sled starts from rest at the top of the frictionless, hemispherical, snow-covered hill shown below.

(a) Find an expression for the sled’s speed when it is at angle $\phi$.

(b) Use Newton’s laws to find the maximum speed the sled can have at angle $\phi$ without leaving the surface.

(c) At what angle $\phi_{\text{max}}$ does the sled “fly off” the hill?

Solution:

(a) When the sled is at a position that relates to angle $\phi$, the height is $y_f = R \cos \phi$. Using conservation of energy with $y_i = R$, we get

\begin{align*}
K_f + U_f &= K_i + U_i \\
\frac{1}{2}mv_f^2 + mgy_f &= \frac{1}{2}mv_i^2 + mgy_i \\
\frac{1}{2}v_f^2 + gR \cos \phi &= 0 + gy_i
\end{align*}

\Rightarrow \quad v_f^2 &= 2gR - 2gR \cos \phi \\
&= 2gR(1 - \cos \phi) \\
\Rightarrow \quad v_f &= \sqrt{2gR(1 - \cos \phi)}
(b) To find the maximum speed we use the free-body diagram:

\[ \sum F = ma = -m \frac{v^2}{R} \]  

(7)

\[ F_N - F_g \cos \phi = -m \frac{v^2}{R} \]  

(8)

The critical speed is just as the normal force equals zero. Factoring the negatives and the masses, and setting \( F_g = mg \) and \( F_N = 0 \),

\[ g \cos \phi = \frac{v_c^2}{R} \]  

(9)

\[ v_c = \sqrt{gR \cos \phi} \]  

(10)

(c) Find the angle when the sled gets air! We can set the speeds from the two previous parts equal to each other to find \( \phi \).

\[ \sqrt{gR \cos \phi} = \sqrt{2gR(1 - \cos \phi)} \]  

(11)

\[ gR \cos \phi = 2gR(1 - \cos \phi) \]  

(12)

\[ \cos \phi = 2 - 2 \cos \phi \]  

(13)

\[ 3 \cos \phi = 2 \]  

(14)

\[ \Rightarrow \phi = \cos^{-1} \left( \frac{2}{3} \right) = 48^\circ. \]  

(15)
12. Truck brakes can fail if they get too hot. In some mountainous areas, ramps of loose
gravel are constructed to stop runaway trucks that have lost their brakes. The combi-
nation of a slight upward slope and a large coefficient of rolling resistance as the truck
tires sink into the gravel brings the truck safely to a halt. Suppose a gravel ramp slopes
upward at 6.0° and the coefficient of rolling friction is 0.40. Use work and energy to find
the length of a ramp that will stop a 15,000 kg truck that enters the ramp at 35 m/s
(≈ 75 mph).

**Solution:**

We’ll identify the truck and the loose gravel as the system. We need the gravel inside
the system because friction increases the temperature of the truck and the gravel.
We will also use the model of kinetic friction and the conservation of energy equation.

\[
K_f + U_f + \Delta E_{th} = K_i + U_i + W_{ext} \tag{1}
\]
\[
0 + U_f + \Delta E_{th} = K_i + 0 + 0 \tag{2}
\]

The thermal energy created by friction is

\[
\Delta E_{th} = f_k(\Delta x) \tag{3}
\]
\[
= (\mu_k F_N)(\Delta x) \tag{4}
\]
\[
= (\mu_k mg \cos \theta)(\Delta x) \tag{5}
\]
\[
= (\mu_k mg \cos \theta)(\Delta x) \tag{6}
\]

Using geometry, the final gravitational potential energy of the truck is

\[
U_f = mgy_f \tag{7}
\]
\[
= mg(\Delta x) \sin \theta \tag{8}
\]

Finally, the initial kinetic energy is simply

\[
K_i = \frac{1}{2}mv_i^2 \tag{9}
\]
Plugging everything into the conservation of energy equation, we get

\[ mg(\Delta x) \sin \theta + \mu_k mg \cos \theta (\Delta x) = \frac{1}{2} m v_i^2 \]  
(10)

\[ g \Delta x (\sin \theta + \mu_k \cos \theta) = \frac{1}{2} v_i^2 \]  
(11)

\[ \Rightarrow \Delta x = \frac{v_i^2}{2g(\sin \theta + \mu_k \cos \theta)} \]  
(12)

\[ = \frac{(35 \text{ m/s})^2}{2 \times (9.8 \text{ m/s}^2) \times (\sin 6^\circ + 0.4 \times \cos 6^\circ)} \]  
(13)

\[ = 124 \text{ m} \]  
(14)

\[ = 0.12 \text{ km}. \]  
(15)
13. The spring shown in the figure below is compressed 50 cm and used to launch a 100 kg physics student. The track is frictionless until it starts up the incline. The student’s coefficient of kinetic friction on the 30° incline is 0.15.

(a) What is the student’s speed just after losing contact with the spring?
(b) How far up the incline does the student go?

\[ k = 80,000 \text{ N/m} \]

\[ m = 100 \text{ kg} \]

\[ 10 \text{ m} \]

\[ 30° \]

Solution:

(a) This is a conservation of energy problem. We want to know the initial velocity immediately after the spring is no longer in contact with the student.

\[ K_f + U_{gf} + U_{sf} + \Delta E_{th} = K_i + U_{gi} + U_{si} + W_{ext} \]  

\[ \frac{1}{2}mv_f^2 + mgy_f + \frac{1}{2}k(\Delta x_f)^2 + 0 = \frac{1}{2}mv_i^2 + mgy_i + \frac{1}{2}k(\Delta x_i)^2 + 0 \]  

\[ \frac{1}{2}mv_f^2 = \frac{1}{2}k(\Delta x_i)^2 \]
\[ \Rightarrow v_f = \sqrt{\frac{k}{m}} \Delta x_i = 14 \text{ m/s} \quad (4) \]

(b) Using the conservation of energy equation again, we can how far the student travels up the incline. Let’s define the \( y \)-height when the student reaches the highest point as \( y_f = \Delta s \sin 30^\circ \). Then we have:

\[
K_f + U_{gf} + U_{sf} + \Delta E_{th} = K_i + U_{gi} + U_{si} + W_{ext} \\
\frac{1}{2}mv_f^2 + mgy_f + \frac{1}{2}k(\Delta x_f)^2 + 0J = \frac{1}{2}mv_i^2 + mgy_i + \frac{1}{2}k(\Delta x_i)^2 + 0J \\
mg\Delta s \sin 30^\circ + \mu_k F_n \Delta s = mgy_i + \frac{1}{2}(\Delta x)^2 \\
(5) \quad (6) \quad (7) \quad (8)
\]

From the sum of the forces in the \( y \)-direction on the incline, the normal force is \( F_n = mg \cos 30^\circ \). Plugging this in, we get

\[
mg\Delta s \sin 30^\circ + \mu_k mg \cos 30^\circ \Delta s = mgy_i + \frac{1}{2}(\Delta x)^2 \\
\Delta smg(\sin 30^\circ + \mu_k \cos 30^\circ) = mgy_i + \frac{1}{2}(\Delta x)^2 \\
\Rightarrow \Delta s = \frac{mgy_i + \frac{1}{2}k(\Delta x)^2}{mg(\sin 30^\circ + \mu_k \cos 30^\circ)} \\
\Delta s = 32.1 \text{ m} \quad (10) \quad (11) \quad (12)\]
14. A hollow sphere is rolling along a horizontal floor at 5.0 m/s when it comes to a 30° incline. How far up the incline does it roll before reversing direction?

**Solution:**

Assume that the hollow sphere is a rigid rolling body and that the sphere rolls up the incline without slipping. Also assume that the coefficient of rolling friction is zero.

The initial kinetic energy, which is a combination of rotational and translational energy, is transformed in gravitational potential energy. Choose the bottom of the incline as the zero of the gravitational potential energy.

Starting from conservation of energy, we have

\[
K_f + U_{gf} = K_i + U_{gi} \tag{1}
\]

\[
\frac{1}{2}Mv_1^2 + \frac{1}{2}I\omega_1^2 + Mgy_1 = \frac{1}{2}Mv_0^2 + \frac{1}{2}I\omega_0^2 + Mgy_0. \tag{2}
\]

Substituting \( v_1 = 0, \, \omega_1 = 0, \, y_0 = 0, \, I = \frac{2}{3}MR^2 \) (appropriate for a hollow sphere), \( \omega_0 = v_0/R \), and solving for the final height, \( y_1 \), we get

\[
0 + 0 + Mgy_1 = \frac{1}{2}Mv_0^2 + \frac{1}{2}\left(\frac{2}{3}MR^2\right)\omega_0^2 + 0 \tag{3}
\]

\[
gy_1 = \frac{1}{2}v_0^2 + \frac{1}{3}R^2\left(\frac{v_0^2}{R^2}\right) \tag{4}
\]

\[
\Rightarrow y_1 = \frac{5v_0^2}{6g} \tag{5}
\]

\[
= 5 \times (5.0 \text{ m/s})^2 \tag{6}
\]

\[
= \frac{6 \times (9.8 \text{ m/s}^2)}{6 \times (9.8 \text{ m/s}^2)} \tag{7}
\]

\[
= 2.126 \text{ m.}
\]
The distance traveled along the incline is

\[ s = \frac{y_1}{\sin 30^\circ} \]

\[ = \frac{2.126 \text{ m}}{0.5} \]

\[ = 4.3 \text{ m.} \]
15. A 10 g bullet traveling at 400 m/s strikes a 10 kg, 1.0-m wide door at the edge opposite the hinge. The bullet embeds itself in the door, causing the door to swing open. What is the angular velocity of the door just after impact?

Solution:

In order to solve this problem, we use the fact that angular momentum is conserved in the collision for the (bullet+door) system. As the bullet hits the door, its velocity $\vec{v}$ is perpendicular to $\vec{r}$. Consequently, the initial angular momentum about the rotation axis, with $r = L$, is

$$L_i = m_B v_B L.$$  \hfill (1)

After the collision, with the bullet in the door, the moment of inertia about the hinges is

$$I = I_{\text{door}} + I_{\text{bullet}}$$  \hfill (2)

$$= \frac{1}{3} m_D L^2 + m_B L^2.$$  \hfill (3)

Therefore, the final angular momentum is

$$L_f = I \omega$$  \hfill (4)

$$= \left( \frac{1}{3} m_D L^2 + m_B L^2 \right) \omega.$$  \hfill (5)
Equating the initial and final angular momentum and solving for $\omega$ we get

\[ L_f = L_i \]

\[ \left( \frac{1}{3} m_D L^2 + m_B L^2 \right) \omega = m_B v_B L \]

\[ \Rightarrow \omega = \frac{3 m_B v_B}{L(m_D + 3 m_B)} \]

\[ = \frac{3 \times 0.010 \text{ kg} \times 400 \text{ m/s}}{1.0 \text{ m} \times (10 \text{ kg} + 3 \times 0.010 \text{ kg})} \]

\[ = 1.2 \text{ rad/s.} \]
16. A 355 mL soda can is 6.2 cm in diameter and has a mass of 20 g. Such a soda can half full of water is floating upright in water. What length of the can is above the water level?

**Solution:**

The buoyant force, $F_B$, on the can is given by Archimedes’ principle.

Let the length of the can above the water level be $d$, the total length of the can be $L$, and the cross-sectional area of the can be $A$. The can is in static equilibrium, so from the free-body diagram sketched above we have

$$
\sum F_y = F_B - F_{G,can} - F_{G,water} = 0 \quad (1)
$$

$$
\rho_{water}A(L - d)g = (m_{can} + m_{water})g \quad (2)
$$

$$
L - d = \frac{(m_{can} + m_{water})}{A\rho_{water}} \quad (3)
$$

$$
\Rightarrow d = L - \frac{(m_{can} + m_{water})}{A\rho_{water}} \quad (4)
$$

Recalling that one liter equals $10^{-3}$ m$^3$ and the density of water is $\rho_{water} = 1000$ kg m$^{-3}$, the mass of the water in the can is

$$
m_{water} = \rho_{water} \left( \frac{V_{can}}{2} \right) \quad (5)
$$

$$
= (1000 \text{ kg m}^{-3}) \times \left( \frac{355 \times 10^{-6} \text{ m}^3}{2} \right) \quad (6)
$$

$$
= 0.1775 \text{ kg}. \quad (7)
$$

To find the length of the can, $L$, we have:

$$
V_{can} = AL \quad (8)
$$

$$
\Rightarrow L = \frac{355 \times 10^{-6} \text{ m}^3}{\pi(0.031 \text{ m})^2} \quad (9)
$$

$$
= 0.1176 \text{ m}. \quad (10)
$$
Finally, substituting everything into equation (4), we get:

\[
    d = 0.1176 \text{ m} - \frac{(0.020 \text{ kg} + 0.1775 \text{ kg})}{\pi \times (0.031 \text{ m})^2 \times (1000 \text{ kg m}^{-3})} \\
    = 0.0522 \text{ m} \\
    = 5.2 \text{ cm.}
\]
17. Water flowing out of a 16 mm diameter faucet fills a 2.0 L bottle in 10 s. At what
distance below the faucet has the water stream narrowed to 10 mm in diameter?

Solution:

We treat the water as an ideal fluid obeying Bernoulli’s equation. The pressure at
point 1 is $p_1$ and the pressure at point 1 is $p_2$. Both $p_1$ and $p_2$ are atmospheric
pressure. The velocity and the area at point 1 are $v_1$ and $A_1$, and at point 2 they
are $v_2$ and $A_2$. Let $d$ be the distance of point 2 below point 1.

First, we use the time it takes to fill a 2.0 L bottle to compute the flow rate, $Q$, the
rate at which water is flowing out of the faucet, remembering that one liter equals
$10^{-3}$ m$^3$:

\[
Q = \frac{(2.0 \text{ L}) \times (10^{-3} \text{ m}^3/\text{L})}{10 \text{ s}} = 2.0 \times 10^{-4} \text{ m}^3/\text{s}. \tag{1}
\]

Next, we use this result to find the velocity, $v_1$, with which the water leaves the
faucet through the $D_1 = 16 \times 10^{-3}$ m diameter aperture (at point 1):

\[
Q = v_1 A_1 \tag{3}
\]

\[
\Rightarrow v_1 = \frac{Q}{A_1} \tag{4}
\]

\[
= \frac{Q}{\pi(D_1/2)^2} \tag{5}
\]

\[
= \frac{2.0 \times 10^{-4} \text{ m}^3/\text{s}}{\pi \times (8 \times 10^{-3} \text{ m})^2} \tag{6}
\]

\[
= 1.0 \text{ m/s}. \tag{7}
\]
Now, from the continuity equation we have

\[ v_2 A_2 = v_1 A_1 \quad (8) \]

\[ \Rightarrow v_2 = v_1 \frac{A_1}{A_2} \quad (9) \]

\[ = v_1 \frac{\pi (D_1/2)^2}{\pi (D_2/2)^2} \quad (10) \]

\[ = v_1 \left( \frac{D_1}{D_2} \right)^2 \quad (11) \]

\[ = 1.0 \text{ m/s} \left( \frac{16 \times 10^{-3} \text{ m}}{10 \times 10^{-3} \text{ m}} \right)^2 \quad (12) \]

\[ = 2.56 \text{ m/s.} \quad (13) \]

Finally, we turn to Bernoulli’s equation to find the height, \( d \):

\[ p_1 + \frac{1}{2} \rho v_1^2 + \rho g y_1 = p_2 + \frac{1}{2} \rho v_2^2 + \rho g y_2 \quad (14) \]

\[ \rho g (y_1 - y_2) = \frac{1}{2} \rho (v_2^2 - v_1^2) \quad (15) \]

\[ gx = \frac{1}{2} (v_2^2 - v_1^2) \quad (16) \]

\[ d = \frac{(v_2^2 - v_1^2)}{2g} \quad (17) \]

\[ = \frac{(2.56 \text{ m/s})^2 - (1.0 \text{ m/s})^2}{2 \times 9.8 \text{ m/s}^2} \quad (18) \]

\[ = 0.283 \text{ m} \approx 28 \text{ cm.} \quad (19) \]
18. A 100 g ice cube at \(-10^\circ C\) is placed in an aluminum cup whose initial temperature is \(70^\circ C\). The system comes to an equilibrium temperature of \(20^\circ C\). What is the mass of the cup?

**Solution:**

There are two interacting systems: aluminum and ice. The system comes to thermal equilibrium in four steps: (1) the ice temperature increases from \(-10^\circ C\) to \(-0^\circ C\); (2) the ice becomes water at \(0^\circ C\); (3) the water temperature increases from \(0^\circ C\) to \(20^\circ C\); and (4) the cup temperature decreases from \(70^\circ C\) to \(20^\circ C\). Since the aluminum and ice form a closed system, we have

\[
Q = Q_1 + Q_2 + Q_3 + Q_4 = 0. 
\]  

(1)

Each term is as follows:

\[
Q_1 = M_{ice} c_{ice} \Delta T 
\]

(2)

\[
= (0.100 \text{ kg}) \times [2090 \text{ J/(kg K)}] \times (10 \text{ K}) 
\]

\[
= 2090 \text{ J}. 
\]

\[
Q_2 = M_{ice} L_f 
\]

(3)

\[
= (0.100 \text{ kg}) \times (3.33 \times 10^5 \text{ J/kg}) 
\]

\[
= 33,300 \text{ J}. 
\]

\[
Q_3 = M_{ice} c_{water} \Delta T 
\]

(4)

\[
= (0.100 \text{ kg}) \times [4190 \text{ J/(kg K)}] \times (20 \text{ K}) 
\]

\[
= 8380 \text{ J}. 
\]

\[
Q_4 = M_{Al} c_{Al} \Delta T 
\]

(5)

\[
= M_{Al} \times [900 \text{ J/(kg K)}] \times (-50 \text{ K}) 
\]

\[
= -(45,000 \text{ J/kg}) M_{Al}. 
\]

Inserting everything into equation (1), we get

\[
43,770 \text{ J} - (45,000 \text{ J/kg})M_{Al} = 0 
\]

\[
\Rightarrow M_{Al} = 0.97 \text{ kg}. 
\]  

(6)
19. A group of rebels wants to invade the castle and they made a trebuchet as their first line of attack. The group optimized their trebuchet so that their projectile can be fired from a great distance to hit the castle wall. Their trebuchet includes a counterweight mass of 931 kg, a moment arm, $R_2 = 2 \text{ m}$, a projectile mass of 7 kg, a moment arm $R_1 = 10 \text{ m}$, an arm mass of 25 kg, and a release angle of $\phi = 90^\circ$ due to a stopper on the trebuchet base that stops the projectile arm from going any further. The pivot point is 8 m above the ground. Find the distance from the castle wall that the trebuchet needs to be placed so that the projectile hits the wall.

**Solution:**

This problem uses the torque created by the difference in the force/moment-arm ratios of the counterweight and projectile to generate an angular acceleration.

There are two torque terms for the trebuchet. The counterweight will create a clockwise rotation (which by convention is negative) and the projectile will create a positive counterclockwise rotation.

Because the projectile is in a cup attached to the arm, the force and the moment arm are perpendicular to each other. For that torque term, $R_2 F_p \sin(\text{angle}) = R_2 F_p$. However, the counterweight and its moment arm are not perpendicular to each other. The figure below shows the angle between them. The torque component for the counterweight is $R_1 F_{CW} \sin \psi$.

Just like the $\sum F = ma$ for linear motion, $\sum \tau = I \alpha$ for rotational motion where $I$ is the moment of inertia of the $R_2$ which we’ll consider to be a rod rotating about it’s end. Alpha is the angular acceleration.
The moment of inertia for a rod rotating about its end is $I = \frac{1}{3}ML^2$. We’re interested in finding the speed of the projectile as it leaves the moment arm, $R_2$. So, we’ll use the length of $R_2$ for $L$ and the mass of that portion of the rod in our calculation. Let the total mass of the rod be $M$ where the mass of the projectile portion of the rod is $N_1$ and the mass of the counterweight portion of the rod is $N_2$. To find $N_1$,

\[
\frac{N_1}{R_1} = \frac{M}{L} \quad (4)
\]

\[
N_1 = \frac{R_1 M}{L} = \frac{10\text{m} \times 25\text{kg}}{12\text{m}} = 20.8\text{kg} \quad (5)
\]

\[
\sum \tau = I\alpha \quad (7)
\]

\[
F_p R_2 - F_{CW} R_1 \sin\psi = I\alpha \quad (8)
\]

\[
m_2gR_2 - m_{CW}gR_1 \sin\psi = \frac{1}{3}MR_2\alpha \quad (9)
\]

Calculate the angular acceleration, $\alpha$,

\[
\alpha = \frac{3g(m_2R_2 - m_1R_1 \sin\psi)}{N_1 R_2^2} \quad (11)
\]
\[ \alpha = \frac{(3 \times 9.8 \text{m/s}^2)(7\text{kg} \times 10\text{m} - 931\text{kg} \times 2\text{m} \sin 143^\circ)}{20.8\text{kg} \times (10\text{m})^2} \tag{12} \]

\[ \alpha = 14.8 \frac{\text{rad}}{s^2} \tag{13} \]

\[ \omega_f \] when the projectile leaves the trebuchet, we need to use kinematics. We’re told in the problem, that the projectile moment arm travels \( \theta = 90^\circ = \frac{\pi}{2} \) radians before the projectile is released.

\[ (\omega_f)^2 = (\omega_i)^2 + 2\alpha \theta \tag{15} \]

\[ (\omega_f)^2 = 2\alpha \theta = 2\alpha \theta \tag{16} \]

\[ \omega_f = \sqrt{2\alpha \theta} = \sqrt{2 \times 14.8 \frac{\text{rad}}{s^2} \times \frac{\pi}{2}} \tag{17} \]

\[ \omega_f = 6.8 \frac{\text{rad}}{s^2} \tag{18} \]

So far, our calculations have been for circular motion as the projectile is moved by the trebuchet. Once the projectile leaves the trebuchet, we need to switch to linear motion.

To find the linear velocity with which the projectile leaves the trebuchet, we convert the angular velocity to linear velocity.

\[ v = \omega_f R_2 \tag{20} \]
\[ v = 6.8 \text{rad/s} \times 10 \text{m} \quad (21) \]
\[ v = 68 \text{m/s} \quad (22) \]
\[ (23) \]

The original description tells us that the projectile travels through an angle of 90° while on the trebuchet. Using this information, we can find the angle at which the projectile leaves the trebuchet and its height.

![Diagram showing the projectile's trajectory.](image)

We need to use the kinematic equations for projectile motion to find the time it takes for the projectile to hit the ground. Then use this time, to calculate the horizontal range of the trebuchet.

\[ v_y = v \sin \theta = 68 \text{m/s} \times \sin 37° = 41 \text{m/s} \quad (24) \]
\[ (25) \]

We’re able to calculate the starting height of the projectile using the figure above. Because the trebuchet travels through 90°, we can use the same triangle for right before the trebuchet is fired and right after the projectile is released. Because these triangles are the same, the height at which the projectile leaves the trebuchet is 16 m.

\[ y_f = y_i + v_y \Delta t - \frac{1}{2}g\Delta t \quad (26) \]
\[ 0 = 16 \text{m} + 41 \text{m/s} \times \Delta t - \frac{1}{2}9.8 \text{m/s}^2 \Delta t \quad (27) \]
\[ (28) \]
Rearranging this equation into the form of a quadratic equation,

\[ 4.9 \text{m/s}^2 \Delta t - 41 \text{m/s} \times \Delta t - 16 \text{m} = 0 \]  
\[ \Delta t = 8.7 \text{sec} \]  
\[ \Delta t = 8.7 \text{sec} \]  

To find the range, we plug this time into the range equation (i.e. no acceleration in the x-direction). We also need to use the initial velocity of the projectile in the x-direction. From the figure above, we can see that the velocity in the x-direction is \( v_x = v \cos 37^\circ \).

\[ x_f = x_i + v_x \Delta t = v \cos 37^\circ \Delta t \]  
\[ x_f = 68 \text{m/s} \times \cos 37^\circ \times 8.7 \text{sec} \]  
\[ x_f = 472 \text{m} \]
Kinematics and Mechanics

\begin{align*}
x_f &= x_i + v_{xi}t + \frac{1}{2}a_xt^2 \\
v_{xf} &= v_{xi} + a_xt \\
v_{xf}^2 &= v_{xi}^2 + 2a_x(x_f - x_i) \\
y_f &= y_i + v_{yi}t + \frac{1}{2}a_yt^2 \\
v_{yf} &= v_{yi} + a_yt \\
v_{yf}^2 &= v_{yi}^2 + 2a_y(y_f - y_i) \\
\theta_f &= \theta_i + \omega_i\Delta t + \frac{1}{2}\omega^2\Delta t^2 \\
\omega_f &= \omega_i + \alpha\Delta t \\
\omega_f^2 &= \omega_i^2 + 2\alpha\Delta\theta \\
s &= r\theta \\
c &= 2\pi r \\
v_t &= \omega r \\
a_r &= \frac{v^2}{r} = \omega^2r \\
v &= \frac{2\pi r}{T}
\end{align*}

Forces

\begin{align*}
\vec{F}_{\text{net}} &= \Sigma \vec{F} = ma \\
\vec{F}_{\text{net}} &= \Sigma \vec{F}_r = ma = m\frac{v^2}{r} \\
F_g &= mg \\
0 < f_k &= < \mu_s F_N \\
f_k &= \mu_k F_N \\
\vec{F}_{\text{AonB}} &= -\vec{F}_{\text{BonA}}
\end{align*}

Momentum

\begin{align*}
\vec{p}_i &= \vec{p}_f \\
\vec{p} &= m\vec{v} \\
(v_{fx})_1 &= \frac{m_1 - m_2}{m_1 + m_2} (v_{ix})_1 \\
(v_{fx})_2 &= \frac{2m_1}{m_1 + m_2} (v_{ix})_1
\end{align*}

Energy

\begin{align*}
K_f + U_{gf} &= K_i + U_{gi} \\
\Delta K + \Delta U + \Delta E_{\text{th}} &= \Delta E_{\text{mech}} + \Delta E_{\text{th}} = \Delta E_{\text{sys}} = W_{\text{ext}} \\
K_f + U_f + \Delta E_{\text{th}} &= K_i + U_i + W_{\text{ext}} \\
K &= \frac{1}{2}mv^2 \\
U &= mgy \\
\Delta E_{\text{th}} &= f_k\Delta s \\
W &= \vec{F} \cdot \vec{d} \\
P &= \vec{F} \cdot \vec{v} \\
P &= \Delta E/\Delta t
\end{align*}

Fluids and Thermal Energy

\begin{align*}
P &= \frac{F}{A} \\
\rho &= \frac{m}{V} \\
Q &= vA \\
v_1A_1 &= v_2A_2 \\
p_1 + \frac{1}{2}\rho v_1^2 + \rho gy_1 &= p_2 + \frac{1}{2}\rho v_2^2 + \rho gy_2 \\
Q_{\text{net}} &= Q_1 + Q_2 + \ldots = 0 \\
Q &= ML_f \text{ for freezing/melting} \\
Q &= Mc\Delta T \\
I &= \sum mr^2 \\
\vec{\tau} &= \vec{r} \times \vec{F} = I\alpha \\
\vec{L} &= m\vec{v} \times \vec{r} \\
\vec{A} \times \vec{B} &= AB\sin\theta \\
K_{\text{rot}} &= \frac{1}{2}I\omega^2 \\
I_{\text{disk}} &= \frac{1}{2}mr^2 \\
I_{\text{hollow sphere}} &= \frac{2}{3}mr^2 \\
I_{\text{door}} &= \frac{1}{3}mr^2 \text{ about its hinges} \\
I_{\text{rod}} &= \frac{1}{12}mr^2 \text{ about its end}
\end{align*}
\[ I_{rod} = \frac{1}{3}mr^2 \] about its center

**Constants**

\[ g = 9.8 \text{ m/s}^2 \]
\[ \rho_{water} = 1000 \text{ kg/m}^3 \]
\[ \rho_{air} = 1.28 \text{ kg/m}^3 \]
\[ c_{ice} = 2090 \frac{J}{kg \text{ K}} \]
\[ c_{water} = 4190 \frac{J}{kg \text{ K}} \]
\[ c_{Al} = 900 \frac{J}{kg \text{ K}} \]
\[ L_f = 333,000 \frac{J}{kg} \text{ ice to water} \]