**Directions:** Answer each question to the best of your ability. You can use your notes, any relevant textbooks, and the web. State explicitly any assumptions that you make, and cite your references. When possible, begin your solutions from first principles (e.g. F= ma, Lagrange’s eqn, etc). You can also ask me questions, but you should not discuss the exam with anyone else.

1. While apple-picking on a beautiful fall day with the Physics Club, James sees an apple falling from a tree. He does some quick calculations in his head (ignoring air resistance) comparing the Lagrangian and Newtonian analysis of the apple’s motion. He realizes with excitement that these two approaches are completely equivalent. Rachel happens to be standing nearby, and James grabs a white board so he can tell Rachel about his findings while providing appropriate visuals. (Needless to say, Jon makes sure that the Physics Club never goes on a trip without an ample supply of whiteboards, markers and erasers.) What does James say/write? Is Rachel convinced?

\[ \text{Newtonian approach: } \]
\[ \sum F = ma \]
\[ x \begin{bmatrix} 0 \\ -F_y \end{bmatrix} = \begin{bmatrix} ax \\ ay \end{bmatrix} \]
\[ \begin{bmatrix} 0 \\ -mg \end{bmatrix} = \begin{bmatrix} ax \\ ay \end{bmatrix} \Rightarrow \begin{bmatrix} ax \\ ay \end{bmatrix} = \begin{bmatrix} 0 \\ -g \end{bmatrix} \]

Integrate w.r.t. \( t \) to get \( \vec{v} \)

\[ \begin{bmatrix} V_x \\ V_y \end{bmatrix} = \begin{bmatrix} \frac{V_{0x}}{C_{1x}} \\ -gt + \frac{V_{0y}}{C_{1y}} \end{bmatrix} = \begin{bmatrix} V_{0x} \\ -gt + V_{0y} \end{bmatrix} \]

Integrate again to get \( \vec{r} \)

\[ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} V_{0x}t + C_{2x} \\ -\frac{1}{2}gt^2 + V_{0y}t + C_{2y} \end{bmatrix} = \begin{bmatrix} x_0 + V_{0x}t \\ y_0 + V_{0y}t - \frac{1}{2}gt^2 \end{bmatrix} \]

For the apple, \( V_{0x} = 0, V_{0y} = 0 \)

\[ \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 - \frac{1}{2}gt^2 \end{bmatrix} \]
Problem 1, Continued

Lagrangian approach

\[ T = \frac{1}{2} m \dot{y}^2 \]
\[ V = mgy \]
\[ L = T - V = \frac{1}{2} m \dot{y}^2 - mgy \]
\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) = m \ddot{y} \]
\[ \frac{\partial L}{\partial y} = -mg \]

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{y}} \right) = m \ddot{y} 
\frac{\partial L}{\partial y} = -mg \]
\[ \frac{d}{dt} \ddot{y} = 0 \Rightarrow -mg - m\ddot{y} = 0 \Rightarrow \ddot{y} = -g \]

We can integrate as in previous page

\[ v_y = -gt + v_{0y} \]
\[ y = -\frac{1}{2} gt^2 + v_{0y} t + y_0 \]

The 2 approaches are equivalent!

Of course, Rachel is convinced.
2. Lauren overhears the conversation and can't help getting drawn in because she thinks that James has misrepresented the apple's motion. "I saw the same apple falling while I was running down that hill toward the tree in question, and it's path looked different from what you described." Give a quantitative description of what Lauren saw, assuming that she was running with a speed of 3 m/s and the hill was sloped 30 degrees above horizontal (her frame of reference is aligned with the incline). Which depiction of the apple's motion is correct?

If we put frame of reference aligned w/incline +
curved on Lauren, then \( \ddot{g} \) looks like

\[
\begin{align*}
\text{to resolve } g \text{ in the } x', y' \text{ coordinates:} \\
\mathbf{a}' &= \begin{bmatrix} -g \cos \theta \hat{x}' + g \sin \theta \hat{y}' & 0 \end{bmatrix} \\

\end{align*}
\]

Integrate to get \( \mathbf{v}' \)

\[
\mathbf{v}' = \int \mathbf{a}' dt = \begin{bmatrix} -g \sin \theta t + \mathbf{v}_{0y}' & g \cos \theta t + \mathbf{v}_{0x}' \end{bmatrix}
\]

To Lauren's:

\[
\begin{align*}
\mathbf{v}_L &= \begin{bmatrix} 3 \hat{x} \end{bmatrix} \\

\end{align*}
\]

\[
\begin{align*}
\mathbf{r}_L &= \begin{bmatrix} x_0 + (3 \text{m/s})t & \frac{1}{2} g \sin \theta t^2 \\
\hat{y} \end{bmatrix} \\

\end{align*}
\]

The 2 depictions are equivalent + equally valid.
3. For their physics project, Angela, Erin, and Patrick decide to analyze the motion of a pendulum in an elevator. They travel to the Empire State Building so they can have a long elevator ride. They explain to the operator that they need to accelerate upward at a rate of 2 m/s², and the operator, who of course loves physics, agrees. They suspend their simple pendulum, which is 30 cm long, from the top of the elevator. The pendulum is set in motion as the elevator accelerates upward, and they record the pendulum’s motion using Logger Pro. Use the Lagrangian formalism to derive the equation(s) of motion for the pendulum.

Next to select an origin that is in an inertial frame of reference.

I selected an origin at the bottom of the building, directly under the elevator.

We can now express the kinetic + potential energies in terms of x + y.

\[
y_1 = y + l \cos \theta, \quad \dot{y}_1 = \dot{y} + l \dot{\sin} \theta \dot{\theta} \\
x_1 = l \sin \theta, \quad \dot{x}_1 = l \cos \theta \dot{\theta}
\]

\[
T = \frac{1}{2} m \dot{x}_1^2 + \frac{1}{2} m \dot{y}_1^2 \\
= \frac{1}{2} m \left( l \cos \theta \right)^2 + \frac{1}{2} m \left( \dot{y} + l \sin \theta \dot{\theta} \right)^2 \\
= \frac{1}{2} m \left( l^2 \cos^2 \theta \dot{\theta}^2 + \dot{y}^2 + 2 \dot{y} \dot{\sin} \theta \dot{\theta} + l^2 \sin^2 \theta \dot{\theta}^2 \right)
\]

\[
V = mg y_1 = mg (y + l \cos \theta)
\]

\[
L = T - V = \frac{1}{2} m l^2 \dot{\theta}^2 + \frac{1}{2} m \dot{y}^2 + m \dot{y} l \sin \theta \dot{\theta} - mg (y + l \cos \theta)
\]

First, I’ll solve the \( \ddot{\theta} \) equation:

\[
\frac{dL}{d\theta} = ml^2 \dot{\theta} + m \dot{y} l \sin \theta \\
\frac{dL}{d\theta} = m \dot{y} l \cos \theta - mg l \sin \theta
\]

\[
\frac{d}{dt} \frac{dL}{d\theta} = ml^2 \ddot{\theta} + m \dot{y} l \sin \theta + m \dot{y} l \cos \theta \dot{\theta}
\]
Continued

\[ \frac{2L}{d\theta} - \frac{1}{dt} \frac{dL}{d\theta} = 0 \]

\[ m \ddot{\theta}\cos \theta - mg \sin \theta - ml^2 \ddot{\theta} - m \dot{y} \sin \theta - m \dot{L} \cos \theta = 0 \]

cancel

\[-mg \sin \theta - ml^2 \ddot{\theta} - m \dot{y} \sin \theta = 0
\]

\[-m \]

\[ l^2 \ddot{\theta} + \left( g + \ddot{y} \right) l \sin \theta = 0 \]

\[ l^2 \]

\[ \ddot{\theta} + \frac{g + \ddot{y}}{l} \sin \theta = 0 \]

\[ \leftarrow \text{this is a s. t. eqn, eqn} \]

\[ y \]

\[ \frac{dx}{dt} = -mg \cos \theta - m \dot{y} \sin \theta - ml \sin \theta \dot{\theta} - ml \cos \theta \dot{\theta}^2 \]

\[ \frac{dx}{dt} - \frac{dL}{d\theta} \frac{d}{dt} \frac{dL}{d\theta} = -mg - m \dot{y} - ml \sin \theta \dot{\theta} - ml \cos \theta \dot{\theta}^2 \]

\[ \frac{dy}{dt} = -g - l \sin \theta \ddot{\theta} - l \cos \theta \dot{\theta}^2 \]

\[ y \text{ eqn} \]
4. Once at the top of the building, they coincidentally meet up with another group of physics students, John, Nate and Andrew, who were attempting to drop pennies from the top of the building as part of their physics project. They forgot to find out how tall the building was, but they knew that they could figure this out if Angela, Erin and Patrick knew how long their elevator ride was. Angela said they didn’t time the ride, but they could probably use the number of pendulum oscillations to estimate the duration of the elevator ride. Can they use the pendulum as a clock? What is the period of its motion? How many oscillations did the pendulum make on its trip to the top of the Empire State Building? Is this the same number that an identical pendulum would make if it was located on the sidewalk outside the building?

Yes, they can use the pendulum as a clock.

\[ \ddot{\theta} + \frac{g + \dot{y}}{l} \sin \theta = 0 \]

\[ \omega^2 = \frac{g + \dot{y}}{l} \]

\[ T = \frac{2\pi}{\omega} = \frac{2\pi \sqrt{l}}{\sqrt{g + \dot{y}}} \]

\[ l = 0.3 \text{ m}, \quad \dot{y} = 2 \text{ m/s} \]

\[ T = 2\pi \sqrt{\frac{0.3 \text{ m}}{11.8 \text{ m/s}^2}} = 1.002 \text{ s} \]

To find the \( \Delta t \) of oscillation, we first need the height of the Empire State building:

\[ h = 381 \text{ m} @ 102^{nd} \text{ floor (wikipedia)} \]

Assuming \( a = 2 \text{ m/s}^2 \):

\[ \dot{y} = g \gamma + \frac{g}{2} \gamma \Delta t^2 \]

\[ \Delta t = \sqrt{\frac{2(291 \text{ m})}{2 \text{ m/s}^2}} = 19.5 \text{ s} \]

\( \Delta t \) pendulum made \( 19.5 \) oscillations.

An identical pendulum on the street would have a period given by

\[ T = 2\pi \sqrt{\frac{l}{g}} = 1.13 \text{ sec} \]

This pendulum would complete

\[ N = \frac{19.5 \text{ s}}{1.13 \text{ s/oscillation}} = 17.7 \text{ oscillations} \]
5. Engineer Shane accidentally left a wrench on a satellite he was working on. The satellite and wrench were launched safely into a geosynchronous orbit over Nairobi, Kenya. However, as the satellite moved into its final orbit, the wrench came loose and fell toward the earth. Shane needs to decide if anyone is in danger of being hit by the wrench as it plummets to the earth (if not, he probably won’t mention it to his boss...). He does a quick estimate of the wrench’s trajectory, ignoring air resistance. What does he find? Should he tell his boss? How long does he have to decide?

The wrench is in orbit with the satellite, and it will stay in orbit even if it becomes dislodged from the satellite!

Shane does not have to tell his boss. No one is in danger, except perhaps the satellite.
6. The solar wind is a stream of ions that are ejected from the Sun. Because Bobby loves space physics so much, he decides to study the motion of one such electron as it encounters the Earth's North magnetic pole for his advanced lab project. Bobby travels to the North Pole to collect some data, and Canes agrees to go with him so he can really experience cold weather. What would they see happen to the electron? (Make some simplifying assumptions and be quantitative.) How would Canes and Bobby detect the presence of this and similar electrons? Can they derive a potential energy function for the electron?

If we look at the N pole, the field lines are coming straight down:

\[ \vec{V}_e = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}, \quad \vec{B}_0 = \begin{bmatrix} 0 \\ 0 \\ -B_0 \end{bmatrix} \]

An electron enters at some arbitrary angle:

\[ \vec{E} = q \vec{E} = 0, \quad \vec{v} \times \vec{B} = q \begin{bmatrix} V_x & V_y & V_z \end{bmatrix} \]

\[ \vec{F} = q \vec{E} + \frac{qem}{m} \begin{bmatrix} -V_y B_0 \\ V_x B_0 \\ 0 \end{bmatrix} \]

\[ \frac{qem}{m} \frac{qem}{m} \begin{bmatrix} -V_y B_0 \\ V_x B_0 \\ 0 \end{bmatrix} \]

\[ \vec{a} = \frac{qem}{m} \frac{qem}{m} \begin{bmatrix} -V_y B_0 \\ V_x B_0 \\ 0 \end{bmatrix} \]

\[ V_x = \text{const}, \text{ so } e^- \text{ undergoes circular motion in } x-y \]

plane + linear motion in \( z \) plane \( \Rightarrow \) spiral

We can estimate the period of the circular motion:

\[ \text{let } V_{xy} = \sqrt{V_x^2 + V_y^2} \]

\[ F_B = F_{\text{centripetal}} \]

\[ \sqrt{V_x B} = \frac{qem}{m} \] \[ \Rightarrow \quad \alpha = \frac{me}{2\pi}, \quad T = \frac{2\pi me}{qem} \]
\[ T = \frac{2\pi n_e}{\nu_B} \Rightarrow \omega = \frac{2n_e}{\nu_B} \Rightarrow \text{radio frequency} \]

You can detect the e⁻'s motion at radio frequencies.

You also might be able to detect optical light (aurora) if e⁻ collides with air molecules.

Can you derive a potential?

Evaluate \( \nabla \times \vec{F}_B \) to find out:

\[
\nabla \times \vec{F}_B = \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
-yB & yB & 0
\end{vmatrix}
\]

\[
= -\frac{\partial}{\partial x}(yB\hat{i}) + \frac{\partial}{\partial y}(-yB\hat{j}) + \frac{\partial}{\partial z}(q\nu_x B)\hat{k} + \frac{\partial}{\partial y}(q\nu_y B)\hat{k}
\]

\[
= qB \left( \frac{\partial \nu_x}{\partial x} + \frac{\partial \nu_y}{\partial y} \right) \hat{k}
\]

\[
= 0 \quad \text{so can derive a potential}
\]